MA 2051: Ordinary Differential Equations

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Chapter 1

Introduction

1.1 Definitions

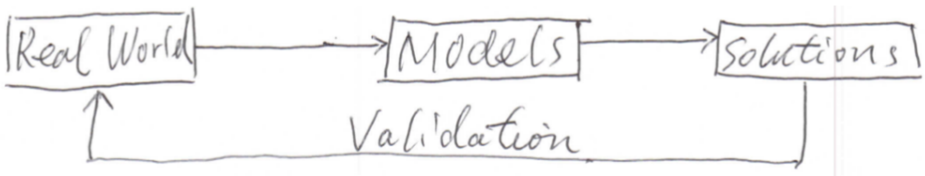


Figure 1.1: A basic overview of ODEs in the real world.

Differential equations is the combination of derivatives and equations. For example, the equation $y'+2y+5x=0$ with the solution $y=y(x)$ is a differential equation with a solution in the form of a function.

Below is a table classifying several differential equations. The order of the differential equation is the highest derivative on any variable. The linearity is determined by the dependent variable (usually y) and if the coefficients are only dependent on a function of the independent variable (usually x).

Order	**Linearity**	**Example**
First Order	Linear	$\frac{dy}{dx}+x^2y=5$
First Order	Non-Linear	$\frac{dy}{dx}+xy^2=5$
Second Order	Linear	$y''+4y'+3y=0.$
Second Order	Non-Linear	$y''+yy'+y=x$

Definition 1.1.1: Homogeneous DiffEQ

For a linear differential equation in *standard form*

$$a_n(x)y^{(n)}(x)+\cdots+a_1(x)y'+a_0(x)y=f(x)$$

the equation is homogeneous if $f(x)=0$. Furthermore, the equation is non-homogeneous if $f(x)\neq 0$.

For example, the equation $y'-y+x=0$ is non-homogeneous because in standard form, the right-hand side is not zero.

Definition 1.1.2: Constant Coefficients

For a linear differential equation which can be represented with constants $a_1, \ldots, a_n$ in *standard form*

$$a_n y^{(n)}(x) + \cdots + a_1 y' + a_0 y = f(x)$$

the equation has *constant coefficients.*

Furthermore, there are two primary forms of an ordinary differential equation. These are described as follows:

1. Implicit Form: $F(x, y, y', y'', \ldots, y^{(n)}) = 0$
2. Explicit Form: $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$

1.2 Verifying Solutions

We begin by defining what a solution to an Ordinary Differential Equation is. For an ordinary differential equation, a solution function $y = y(x)$ satisfies the ODE when y and its higher derivatives are plugged into $F(x, y, y', y'', \ldots, y^{(n)}) = 0$. We break down this process into three simple steps.

1. Using the given solution $y = y(x)$, calculate its higher derivatives as present in the ODE.
2. Plug into the ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$.
3. Solve and check.

We show these steps with an example.

Example: Verifying Solution

Verity $y(x) = \sin(x) - \cos(x) + 1$ is a solution of $y'' + y = 1$.
First, we find the higher derivatives of y. We know that

$$\begin{aligned} y' &= \cos(x) + \sin(x) \\ y'' &= -\sin(x) + \cos(x) \end{aligned}$$

So,

$$\begin{aligned} y'' + y' &= (-\sin(x) + \cos(x)) + (\sin(x) - \cos(x) + 1) \\ &= 1 \end{aligned}$$

Therefore, we have verified the ODE.

Example: An Implicit Case

Show that $x^2 + y^2 = C$ is a solution to $y'y + x = 0$.
First, we implicitly differentiate $x^2 + y^2 = C$ to get $y' = -\frac{x}{y}$.
Plugging this into the ODE, we get, $-\frac{x}{y} \cdot y + x = 0$. This equation holds, that means we have verified the solution.

We now introduce *initial value problems.* To put it simply, an initial value problem (IVP for short), is an ordinary differential equation $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ with an initial value $y(x_0) = y_0$ to satisfy. Therefore, a solution to this IVP is a function $y = y(x)$ such that it satisfies both the ODE *and* the initial value condition.

For example, a first-order differential equation could have the following IVP.

$$\text{IVP}: \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \qquad \text{(Initial Condition)} \end{cases}$$

How do we know if a solution to an IVP is unique or even exists? Let us introduce the next theorem.

Theorem 1.2.1: Picard's Theorem (Existence / Uniqueness)

For a first-order differential equation IVP given by

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

and the functions f and $\frac{\partial f}{\partial y}$ are continuous over a rectangular region R that contains (x_0, y_0) where $R : \{(x,y) : a < x < b, c < y < d\}$. Under these conditions, there **exists a unique solution** $y = \phi(x)$ on some interval $[x_0 - h, x_0 + h]$ for some h.

Chapter 2

First-Order Differential Equations

2.1 Integrating Factors

For first-order differential equations in the form $a_0(x)y' + a_1(x)y = a_2(x)$, we can simplify this to the form $y' + p(x)y = q(x)$. To solve an ordinary differential equation in this form, we first find an *integrating factor*.

Let's find a function $\mu(x)$ so that we can cleverly use the product rule.

$$\begin{aligned}
\mu(x)(y' + p(x)y) &= (\mu(x)y)' \\
\mu(x)y' + \mu(x)p(x)y &= \mu'(x)y + \mu(x)y' \\
\mu(x)p(x)y &= \mu'(x)y \\
\mu(x)p(x) &= \mu'(x)
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= p(x) \\
\int \frac{\mu'(x)}{\mu(x)} dx &= \int p(x)dx
\end{aligned}$$

Therefore, we arrive at the following definition.

Definition 2.1.1: Integrating Factor ($\mu(x)$)

For an ordinary differential equation in the form $y' + p(x)y = q(x)$, the integrating factor, $\mu(x)$ can be represented as

$$\mu(x) = e^{\int p(x)dx}$$

So how does this help? Note that if we multiply both sides of the differential equation by the integrating factor, we get

$$\mu(x)(y' + p(x)y) = q(x)$$
$$(\mu(x)y)' = \mu(x)q(x)$$
$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)dx$$

And we have arrived at the solution y. Note that this *only applies* to differential equations in the form $y' + p(x)y = q(x)$.

Let us begin with an introductory example.

Example: Integrating Factor

Solve the differential equation $y' - 2xy = e^{x^2}$.
First, we note that $p(x) = 2x$, so $\mu(x) = e^{\int -2x dx} = e^{-x^2}$. So, we can now find the solution.

$$(e^{-x^2}y)' = e^{-x^2}e^{e^2}$$
$$y = e^{x^2} \int e^{-x^2} \cdot e^{x^2} dx$$
$$= e^{x^2}(x + C)$$
$$= xe^{x^2} + Ce^{x^2}$$

Now we show an example with an initial value problem, otherwise known as an IVP.

Example: IVP

Solve the IVP where $(1 + e^x)y' + e^x y = 0$ where $y(0) = 1$.
First, we transform this equation into the form that is friendly for integrating factors. So, $y' + \frac{e^x}{1+e^x}y = 0$. We know that $p(x) = \frac{e^x}{1+e^x}$, so $\mu(x) = \exp(\frac{e^x}{1+e^x})$. By u-substitution, we find that $\mu(x) = e^{\ln(1+e^x)} = 1 + e^x$. So, we can now solve this differential equation.

$$((1 + e^x)y)' = (1 + e^x) \cdot 0$$
$$y = \frac{1}{1 + e^x} \int 0dx$$
$$= \frac{C}{1 + e^x}$$

Next, we apply our initial condition to solve for c. So,

$$1 = \frac{C}{1 + e^0}$$
$$2 = C$$

So, the solution to the IVP is

$$y = \frac{2}{1 + e^x}$$

2.2 Separable Equations

Otherwise known as the Separation of Variables method, this method is useful for the simplest first-order differential equations.

Definition 2.2.1: SOV Method

For some differential equation $\frac{dy}{dx} = f(x) \cdot g(y)$, we can solve this differential equation by transforming it into the form

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Note that any function that cannot be expressed as a product of a function of x and a function of y is called *non-separable*. Let us begin with another example.

Example: Basic Example

Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.
We perform the following operations.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{x}{y} \\ \int y dy &= \int -x dx \\ \frac{y^2}{2} &= \frac{-x^2}{2} + C \\ x^2 + y^2 &= C \end{aligned}$$

We continue with a second example, sightly harder this time with the algebraic manipulation.

Example: Harder Example

Solve the differential equation $\frac{dy}{dx} = y - y^2$.
We perform the following operations.

$$\begin{aligned} \frac{dy}{dx} &= y^2 - y \\ \int \frac{1}{y(y-1)} dy &= \int 1 dx \\ \int \frac{A}{y} + \frac{B}{y-1} dy &= x + C \\ \int \frac{-1}{y} + \frac{1}{y-1} dy &= x + C \\ -\ln|y| + \ln|y-1| &= x + C \\ \ln|\frac{y-1}{y}| &= x + C \\ \frac{y-1}{y} &= \pm Ce^x \\ y &= \frac{1}{1 - Ce^x} \end{aligned}$$

2.3 Growth and Decay

For natural processes, the growth or decay of some quantity is often proportional to it's current state of quantity. That is, let p be the quantity of something, that k be the "rate of growth/decay." Then, we are investigating the IVP

$$\begin{aligned} \frac{dp}{dt} &= kp \\ p(t_0) &= p_0 \end{aligned}$$

The solution to this IVP is $p(t) = p_0 e^{kt_0}$. This can be applied to virtually any growth or decay model. We show the derivation below.

$$\begin{aligned} \frac{dp}{dt} &= kp \\ \frac{1}{p}dp &= kdt \\ \int \frac{1}{p}dp &= \int kdt \\ \ln|p| &= kt + C \\ p &= Ce^{kt} \\ p &= p_0 e^{k(t-t_0)} \end{aligned}$$

Example: Growth Model

Assume $k = 0.02$. How long does it take for $p(t)$ to double?
We begin with the following steps.

$$\begin{aligned} 2p_0 &= p_0 e^{0.02(t-t_0)} \\ 2 &= e^{0.02(t-t_0)} \\ \ln(2) &= 0.02(\Delta t) \\ \Delta t &\approx 34.7 \end{aligned}$$

Example: Decay Model

It is known that the chemical element carbon-14's half-life is 5600 years. Find it's k value.
Therefore, we can perform the following.

$$\begin{aligned} 0.5p_0 &= p_0 e^{5600k} \\ 0.5 &= e^{5600k} \\ \ln(0.5) &= 5600k \\ k &\approx -0.0001238 \end{aligned}$$

Example: Periodic Savings

Let m be the amount in a savings account and r be the interest rate compounding continuously. Assume we add some constant yearly deposit into our account d. That is,

$$\frac{dm}{dt} = rm + d$$
$$m(t_0 = 0) = m_0$$

We solve this differential equation with integrating factors. Assume $p(t) = -r$ and $q(t) = d$. So,

$$\mu(t) = \exp\left(\int -rdt\right) = e^{-rt}$$

So, we can now solve the differential equation.

$$(e^{-rt} \cdot m)' = e^{-rt} \cdot d$$
$$m(t) = e^{rt} \int de^{-rt} dt$$
$$m(t) = \frac{-d}{r} e^{rt} \left(e^{-rt} + C\right)$$
$$m(t) = Ce^{rt} - \frac{d}{r}$$

Solving the IVP where $m(0) = m_0$, we can show that

$$m_0 = C - \frac{d}{r}$$

and therefore,

$$C = m_0 + \frac{d}{r}$$

So, we conclude that

$$m(t) = \left(m_0 + \frac{d}{r}\right) e^{rt} - \frac{d}{r}$$

This solution is often called *periodic savings* for continuous interest rates.

2.4 Cooling and Heating Phenomena

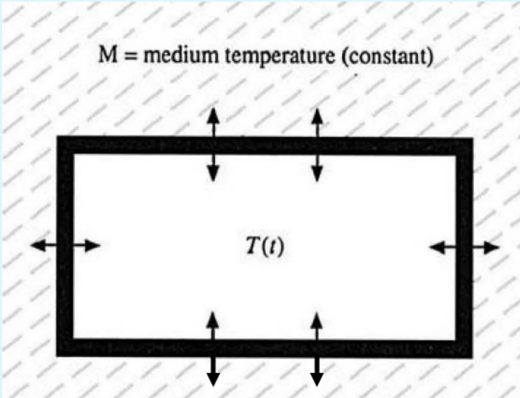
Specifically, we are going to explore Newton's Cooling Law. This law applies to systems with changing temperatures between an outside environment and an interior envirionment.

Theorem 2.4.1: Newton's Law of Cooling

The rate of change, $\frac{dT}{dt}$, in the temprature T of a body placed in a medium of temperature M is proportional to the difference in temperatures. That is,

$$\frac{dT}{dt} = -k(T - M)$$

for a constant of proportionality $k > 0$, also known as a time constant where time constant $= \frac{1}{k}$.



For the IVP where $\frac{dT}{dt} = -k(T-M)$ and $T(0) = T_0$, we can derive the solution using separation of variables.

$$\frac{dT}{dt} = -k(T-M)$$
$$\int \frac{dT}{T-M} = \int -kdt$$
$$\ln|T-M| = -kt + C$$
$$T - M = Ce^{-kt}$$

Now, we plug in the initial value to arrive at the solution below. Note that the constant $+M$ is called the steady state.

$$T(t) = (T_0 - M)e^{-kt} + M$$

We begin with an example.

Example: Furnace

A furnace broke down at 12:00am ($t = 0$), and room temperature was 70 degrees F. The outside temperature was 20 degrees F. At 2:00am, $t = 2$, the building temperature was 20 degrees F.
a) Find k.
The IVP we are trying to solve is $\frac{dT}{dt} = -k(T-20)$ where $T(0) = 70$. So, we arrive at the solution equation

$$T(t) = 50e^{-kt} + 20$$

Plugging in the values $2, 50$, we find that

$$50 = 50e^{-2k} + 20$$

$$k = -\frac{\ln(0.6)}{2} = 0.255$$

b) When will T be 40 degrees F?
We can solve this by now plugging in k.

$$40 = 50e^{-0.255t} + 20$$

$$t = 3.592 = 3.35\text{am}$$

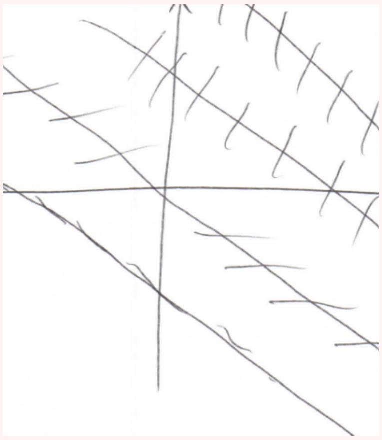
2.5 Directional Field's and Euler's Method

We know that the derivative of a function $\frac{df}{dx}$ can be interpreted as the slope of a tangent line to a function $f(x)$. Therefore, for every point (x, y), we can calculate the slope $m = \frac{dy}{dx} = \tan(\theta)$ with angle θ from the origin. Therefore, we call this θ the direction, and with the derivative evaluated at all (x, y), we find that we can create a *directional field.*

How can we draw a directional field by hand? A good way to approach this is to look at when $\frac{dy}{dx}$ equals some constant. Let's take a look at the following example.

Example: Drawing Directional Field

Let $\frac{dy}{dx} = x + y$. Find the directional field.
We first let $x + y$ equal some constants. Let's first try, $x + y = 1$, so along the line of $x + y$, the slope along that line will be 1 on the coordinate plane. For $x + y = c$ and some constant c, the directions along that line will have slope c. So, we arrive at the following image.



What if we wanted a numerical approximation for an IVP? One way to do this is Euler's method, where we take advantage of the linearity property of tangent lines. On each step from a known point, we use the tangent line as an approximation.

Theorem 2.5.1: Euler's Method

Let there be some function $y(x)$ such that $\frac{dy}{dx} = f(x, y)$. Then, for some step k, we can find it recursively with the formula,

$$y_{k+1} = y_k + f(x_k, y_k) \cdot (x_{k+1} - x_k)$$

To put it in more simple terms, we need what our step size is ($\Delta x = x_{k+1} - x_k$), our number of steps n, and the interval $[a, b]$ for which we are performing this method on. Let us demonstrate this with an example.

Example: Euler's Method Example

Let $\frac{dy}{dx} = x + y$ and $y(0) = 1$. Approximate $y(1)$ using Euler's Method with step size 0.25.
We will construct a table to make our lives easier.

(x_k, y_k)	y'_k	$y_{k+1} = y_k + y'_k \cdot \Delta x$
$(0, 1)$	$y'_0 = 1$	$y_1 = 1 + 1 \cdot 0.25$
$(0.25, 1.25)$	$y'_1 = 1.5$	$y_2 = 1.25 + 1.5 \cdot 0.25$
$0.5, 1.625$	$y'_2 = 2.125$	$y_3 = 1.625 + 2.125 \cdot 0.25$
$0.75, 2.15625$	$y'_3 = 2.90625$	$y_4 = 2.15625 + 2.90625 \cdot 0.25$

So, we can conclude that $y(1) \approx 2.8828125$.

Chapter 3

Second-Order Linear Equations

3.1 Introduction

First and foremost, let us start where some definitions. A second-order differential equation is in the form,

$$y'' + p(x)y' + q(x)y = f(x)$$

An IVP for a second order differential equation would therefore have two parameters, $y(t_0) = y_0$ and $y'(t_0) = v_0$.

This differential equation is said to be homogeneous when $f(x) = 0$ and nonhomogeneous otherwise. A general solution has two main parts, which are a homogeneous and a nonhomogeneous part. Beyond just defining the solution, we can guarantee a unique solution to each second order differential equation is $p(x)$, $q(x)$, and $f(x)$ are continuous over an interval about t_0.

Theorem 3.1.1: General Solution Structure

Given a second order differential equation in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

The *general solution* of the differential equation will be in the form

$$y = cy_h + y_p$$

where y_h is the homogeneous solution to $y'' + p(x)y' + q(x)y = 0$ and y_p is the particular solution to $y'' + p(x)y' + q(x)y = f(x)$.

In summary, you *only need* a single particular solution in your general solution such that it satisfies the particular condition.

Proof: General Solution

Let y_p be a particular solution and y_g be another solution to $y'' + p(x)y' + q(x)y = f(x)$. Therefore, we can say that $y'' + p(x)y' + q(x)y = 0$.

$$y_g = (y_g - y_p) + y_p$$

Therefore, we can show that $(y_g - y_p)$ is a homogeneous solution. Therefore, we can substitute this into $y'' + p(x)y' + q(x)y = 0$ such that

$$
\begin{aligned}
(y_g - y_p)'' + p(x)(y_g - y_p)' + q(x)(y_g - y_p) &= (y_g'' + p(x)y_g' + q(x)y_g') \\
&\quad - (y_p'' + p(x)y_p' + q(x)y_p) \\
f(x) - f(x) &= 0
\end{aligned}
$$

Therefore, we have proved that all solutions can be expressed as a linear combination of y_g and y_p.

So, this leads us to the Superposition Principle due to the linearity of solutions of second order differential equations. We introduce this as follows.

Theorem 3.1.2: Superposition Principle

If y_1 and y_2 are solutions of $y'' + p(x)y' + q(x)y = 0$, then any linear combination of $c_1y_1 + c_2y_2$ is also a solution.

3.2 Fundamental Solutions of Homogeneous Eq

It is difficult to apply techniques from first-order differential equations to equations of the second-order. However, we can show that a solution of a second-order homogeneous equation will usually result in a linear combination of two separate solutions. First, we introduce the concept of linear independence.

Definition 3.2.1: Linear Independence

Two functions y_1 and y_2 are linearly independent if and only if the only c_1 and c_2 that satisfy

$$c_1y_1 + c_2y_2 = 0$$

is $c_1 = 0$ and $c_2 = 0$. If there exists such c_1 and c_2 other than 0, then we say these functions are linearly dependent.

An easier way to check linear independence is by using the Wronskian, which we define as follows.

Definition 3.2.2: Wronskian

The Wronskian of two functions y_1 and y_2 is defined as

$$
\begin{aligned}
W(y_1, y_2) &= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \\
&= y_1y_2' - y_2y_1'
\end{aligned}
$$

We can prove linear independence using the following theorem.

Theorem 3.2.3: Linear Independence

For any y_1 and y_2, if $W(y_1, y_2) \neq 0$, y_1 and y_2 are linearly independent.

Therefore, based on the previous Theorem, for two solutions y_1 and y_2 of a homogeneous equation $y'' + p(x)y' + q(x)y = 0$, then we can determine independence from the Wronksian. Furthermore, we can say that *all solutions* of a homogeneous equation can be expressed as a linear combination of y_1 and y_2 if they are linearly independent. Below, we define the solutions to a homogeneous differential equation is

Definition 3.2.4: Solutions to Homogeneous Eq

For two homogeneous solutions y_1 and y_2, if they are linearly independent and $W(y_1, y_2) = 0$, then any solution of the differential equation can be expressed as

$$y = c_1 y_1 + c_2 y_2$$

where $\{y_1, y_2\}$ is called the fundamental solution set of the homogeneous differential equation.

In the next section, we discuss solutions of nonhomogeneous differential equations, which contain an extra component: the particular solution y_p.

3.3 Homogeneous Eq with Constant Coefficients

3.3.1 Real, Distinct Roots

Given a homogeneous second order differential equation in the form

$$ay'' + by' + cy = 0$$

we can guess that a solution for one of the two parts is in the form e^{rx}. Plugging this in, we get the *characteristic equation*

$$ar^2 + br + c = 0$$

Solving for r, we get two distinct solutions r_1 and r_2 called *eigenvalues* where $e^{r_1 x}$ and $e^{r_2 x}$ are called eigenfunctions. Therefore, the general solution can be written as a linear combination in the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Let us begin with an example.

Example: Real, Distinct Roots

Find the solutions to $y'' + 5y' - 6y = 0$.
First, we start off with the characteristic polynomial

$$r^2 + 5r - 6 = 0$$

with solutions $r_1 = -6$, $r_2 = 1$. Therefore, the eigenfunctions are e^{-6x} and e^{1x} such that the general solution can be represented as

$$y(x) = c_1 e^{-6x} + c_2 e^{x}$$

3.3.2 Repeated Root

If the characteristic polynomial from above gives us a repeated root where $r = r_1 = r_2$, our solution must be in the following form.

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

This can be found by "guessing" whether or not xe^{rx} could work as a solution after finding our first solution in the form e^{rx}. We show this solution structure with an example.

Example: Repeated Root

Find the solutions to differential equation $y'' + 4y' + 4y = 0$.
First, we write down the characteristic polynomial $r^2 + 4r + 4 = 0$ with repeated root $r = -2$. Therefore, our solution can be written in the form

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

3.3.3 Imaginary Roots

If our eigenvalues r_1 and r_2 are imaginary in the form $\pm bi$, our eigenfunctions in the form $ce^{\pm bi}$ will have little to no meaning. However, we can use Euler's Formula, which states that

$$e^{ix} = \cos x + i \sin x$$

Therefore, we can plug in our solutions to get the two equations

$$e^{bix} = \cos(bx) + i \sin(bx)$$

and

$$e^{-bix} = \cos(bx) - i \sin(bx)$$

Therefore, we can express the following functions

$$\cos(bx) = \frac{e^{bix} + e^{-bix}}{2}$$

$$\sin(bx) = \frac{e^{bix} - e^{-bix}}{2i}$$

which are linearly independent. Therefore, $\sin(x)$ and $\cos(x)$ are solutions to this equation.

Example: Imaginary Roots

Solve the equation $y'' + 4y = 0$ with initial value conditions $y(0) = 1$ and $y'(0) = -1$
With the characteristic equation $r^2 + 4 = 0$, we get repeated imaginary roots $r_1, r_2 = \pm 2i$. Therefore, our solution is in the form

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Solving for our initial conditions, we get that $c_1 = 1$ and $c_2 = -\frac{1}{2}$. So our solution to the IVP is

$$y(x) = \cos(x) - 0.5 \sin(2x)$$

Complex Roots

For roots $r_1, r_2 = a \pm bi$, we will get the general solution in the form

$$y(x) = c_1 e^{ax} \cos(bx) + c_2 e^{ax} \sin(bx)$$

Example: Complex Roots

Find the general solution to $y'' + 2y' + 2y = 3$.
We first find the characteristic equation $r^2 + 2r + 2 = 0$ with complex roots $r_1, r_2 = -1 \pm 2i$. So, our homogeneous solution will be in the form

$$y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$$

However, since our equation is not homogeneous, we must find a particular solution. Note that $y_p = 5$ works as a particular solution, therefore, our solution can be expressed as

$$y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x) + 5$$

3.4 Non-Homogeneous Solutions

Recall that a second-order non-homogeneous differential equation is in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

To introduce how we obtain solutions, we provide a general framework for the superposition principle for a family of differential equations with particular part $f(x)$.

Theorem 3.4.1: Superposition Principle

1. **Scalar Property: for non-homogeneous part $f(x)$ which gives solution y, then $kf(x)$ gives ky.**
2. **Additive Property**: for non-homogeneous part f_1 and f_2 which give solutions y_1 and y_2 respectively, then $f_1 + f_2$ gives $y_1 + y_2$ as a solution.

This principle allows us to find particular solutions to different parts of a non-homogeneous separately and combine them at the end of the full particular solution. Note that, then, the general solution to a non-homogeneous differential equation can be written in the form

$$y(x) = c_1 y_1 + c_2 y_2 + y_p$$

for homogeneous solutions y_1, y_2 and particular solution y_p. This gives us a two-step process for solving non-homogeneous equations.

1. Find the homogeneous solutions y_1, y_2 to $y'' + p(x)y' + q(x)y = 0$.
2. Find the particular solution y_p to $y'' + p(x)y' + q(x)y = f(x)$.
3. Combine in the form above for the general solution.

3.4.1 Method of Undetermined Coefficients

For the method of undetermined coefficients, our differential equation will be in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where we will need to guess for some form of y, which will *work out* in finding the particular solution. Below, we will go over the main types of particular solutions we will have to guess-and-check for.

n'th Degree Polynomial

For a particular part $f(x)$ which is an n'th degree polynomial, we can "guess" a particular solution in the form

$$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$$

Let's start with an example of a linear polynomial.

Example: Linear Polynomial

Find the particular solution to the differential equation $y'' + 2y' + y = x$.
First, we identify $f(x) = x$, so we guess a particular solution in the form $Ax + B$. Plugging this in, we get

$$2(Ax + B)' + (Ax + B) = x$$

which shows us that $A = 1$ and $B = -2$. Therefore,

$$y_p = x - 2$$

We can find the remaining homogeneous solutions with their characteristic equation from previous sections.

Example: Quadratic Polynomial

Find the particular solution to differential equation $y'' + 2y' + y = 2x^2$.
First, we guess that particular solution $f(x)$ can be expressed in the form $Ax^2 + Bx + C$.
Therefore, if we plug this particular solution into the differential equation, we get

$$2A + 2(2Ax + B) + (Ax^2 + Bx + C) = 2x^2$$

Therefore, we get from solving a systems of equations that $A = 2$, $B = -8$, and $C = 12$. So, our particular solution is

$$y_p = 2x^2 - 8x + 12$$

Exponential Functions

For exponential functions with particular parts ae^{bx}, we can guess some particular solution in the form

$$y_p = Ae^{bx}$$

Let us try out an example.

Example: Exponential Function

Find solutions to the differential equation $y'' + 3y' + 2y = 2e^{3x}$.
First, we guess a particular solution in the form Ae^{3x}. So, if we plug this into the differential equation, we get

$$9Ae^{3x} + 9Ae^{3x} + 2Ae^{3x} = 2e^{3x}$$

where $A = \frac{1}{10}$. Therefore, our particular solution is

$$y_p = \frac{1}{10}e^{3x}$$

After we find our particular solutions, however, make sure to find the general solutions to make sure the particular solution is found in the homogeneous solution. Otherwise, we may need to use the form $y_p = Ax^n e^{bx}$ for the particular solution (usually $n = 1$) or even some higher degree of x.

Trigonometric Functions

For any particular functions $f(x)$ in the form $a\sin(bx) + c\cos(bx)$, we can guess a solution in the form $y_p = A\sin(bx) + B\sin(bx)$. However, if we have *different* b values, we will have to find *two* particular solutions by the Superposition Principle.

Example: Sine Function

For the differential equation in the form $y'' - 3y' + 2y = \sin(2x)$.
We can first guess a particular solution in the form $y_p = A\sin(bx) + B\sin(bx)$. Therefore, we can plug this into the equation to get

$$(-4A\sin(2x) - 4B\cos(2x)) - 3(2A\cos(2x) - 2B\sin(2x) \\ + 2(A\sin(2x) + B\cos(2x)) = \sin(2x)$$

Simplifying and solving the system, we get $A = -\frac{1}{20}$ and $B = \frac{3}{20}$
such that $y_p = -\frac{1}{20}\sin(2x) + \frac{3}{20}\cos(2x)$

Similar to the case of exponential functions, if $\sin(x)$ or $\cos(x)$ is a part of the homogeneous solution, we must multiply by a power of x^n, usually at most $n = 1$.

A general method for the Method of Undetermined Coefficients is described as follows.s

Definition 3.4.2: Alternative Method

For $ay'' + by' + cy = f(x)$, if $\{f(x), f'(x), f''(x), \cdots f^{(n)}(x)\}$ is a finite set, then the undetermined coefficients method can work where we assume a particular solution in the form

$$y_p = a_1 f + a_2 f' + a_3 f'' + \cdots a_n f^{(n)}$$

However, this will not work for functions such as $\ln(x)$, where the derivatives never have an end.

3.5 Variation of Parameters

For a general case solution to a non-homogeneous differential equation with non-homogeneous part $f(x)$, we can use the following variation of parameters method.

Definition 3.5.1: Variation of Parameters

If y_1 and y_2 are the two homogeneous solutions (or form the fundamental solution set) of

$$y'' + p(x)y' + q(x)y = 0$$

Then, we can find the particular solution as

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$$

where

$$v_1(x) = \int \frac{-y_2 f(x)}{W(y_1, y_2)} dx$$

$$v_2(x) = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

This can be derived from attempting to find a particular solution as some combination of y_1 and y_2 and making some assumptions to clean up the derivatives. Let us show a quick example of the variation of parameters method.

Example: Variation of Parameters

Find the solution to $y'' + y = \sec(x)$.
First, we find the homogeneous solutions (fundamental solution set) to $y'' + y = 0$. We first write the characteristic equation as follows.

$$r^2 + 1 = 0$$

with solutions $r = \pm i$. Therefore, our fundamental solution set consists of $y_1 = \sin(x)$ and $y_2 = \cos(x)$. So, our general solution is in the form

$$y(x) = c_1 \sin(x) + c_2 \cos(x) + y_p$$

Now it is time to find the particular solution. Note that our non-homogeneous part $f(x) = \sec(x)$ does not fall clearly into any of the categories for the Method of Undetermined Coefficients. So, we use the Variation of Parameters method to find a particular solution in the form

$$y_p = v_1(x)y_1 + v_2(x)y_2$$

We now find $v_1(x)$ and $v_2(x)$ using the formulas above. We first find the Wronskian to be

$$W(\sin(x), \cos(x)) = -\sin^2(x) - \cos^2(x) = -1$$

$$v_1(x) = \int (-1) \cdot -\cos(x) \cdot \sec(x) dx$$
$$\int 1 dx = x$$

$$\begin{aligned} v_2(x) &= \int (-1) \cdot \sin(x) \cdot \sec(x) dx \\ &= -\int \frac{\sin(x)}{\cos(x)} dx \\ &= \int \frac{1}{u} du \\ &= \ln(\cos(x)) \end{aligned}$$

So, our particular solution is

$$y_p = x \cdot \sin(x) + \ln(\cos(x)) \cdot \cos(x)$$

So, our general solution is

$$y(x) = c_1 \sin(x) + c_2 \cos(x) + x \sin(x) + \cos(x) \ln(\cos(x))$$

3.6 Physical Models

3.6.1 Spring-Mass System

When we have a mass hanging by a spring, we can say that the force F can be expressed using Hooke's Law with y representing the distance stretched from the initial stretch distance.

$$F = -ky$$

Using Newton's Second Law and assuming no air resistance and no fatigue of the spring, we can substitute $F = ma = my''$ to get our differential equation

$$my'' = -ky$$

Example: Spring-Mass System

A 2 kg weight is suspended on a spring in the air. The spring is extended 10cm to reach a balancing point. Solve the IVP and find k.
First, we find k. We create a free-body diagram to find that

$$F = 2 \cdot 9.8 = k \cdot 0.1$$

and $k = 196 \frac{N}{m}$. Therefore, we can express our Spring-Mass system as

$$2y'' + 196y = 0$$

with initial value $y(0) = y_0$ and $y'(0) = 0$ as the spring is not moving in the beginning. To solve this IVP, we first write the characteristic equation

$$r^2 + 98 = 0$$

with $r = \pm 9.9i$. Therefore, we get general solutions $y_1 = \cos(9.9t)$ and $y_2 = \sin(9.9t)$ with general solution in the form

$$y_g(t) = c_1 \cos(9.9t) + c_2 \sin(9.9t)$$

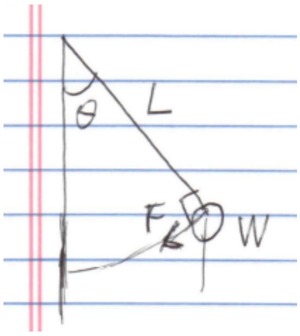
Now we solve for the initial value. Plugging in $y(0)$, we get $y(0) = c_1 = y_0$. Furthermore, when we plug in $y'(0)$, we get $y'(0) = 0 - c_2 \ = 0$. Therefore, our solution to the IVP is

$$y(t) = y_0 \cos(9.9t)$$

To find the period of a spring-mass system, set the part inside the sinusoidal function equal to 2π. For the example above, we will need to solve the equation $9.9T = 2\pi$ to find period T in seconds.

3.6.2 Pendulum Models

Below is an image of what a pendulum model looks like.



We can set up our differential equation to be

$$F = ma = mS'' = mL\theta''$$

Furthermore, from our simple harmonic motion equations, we also have

$$F = \omega \sin(\theta) = mg \sin(\theta)$$

Therefore, we can set up our differential equation as

$$mL\theta'' = -mg \sin(\theta)$$

We can then approximate $\sin(\theta)$ as θ for small enough θ. Therefore, we arrive at our IVP

$$L\theta'' + g\theta = 0$$

Example: Harmonic Motion with Pendulum

Solve the IVP with the above differential equation and initial conditions $\theta(0) = 0.1$ and $\theta'(0) = 0$. First, we write the characteristic equation

$$r^2 + \frac{g}{L} = 0$$

with $r = \pm\sqrt{\frac{g}{L}}i$. Therefore, our general solution is in the form

$$\theta(t) = c_1 \cos(\sqrt{\frac{g}{L}}t) + c_2 \sin(\sqrt{\frac{g}{L}}t)$$

Plugging in our initial conditions, we get $c_1 = 0.1$ and $c_2 = 0$. Therefore, we arrive at solution

$$\theta(t) = 0.1 \cos(\sqrt{\frac{g}{L}}t)$$

To find the period of a pendulum, set the part inside the sinusoidal function equal to 2π. For the example above, we will need to solve the equation $0.1T = 2\pi$ to find period T in seconds.

3.6.3 Damping Models

For a model with a damping factor k_d, we have differential equation

$$F = my'' = -k_f y - k_d y' + f(t)$$

with spring factor $k_f > 0$, damping factor $k_d > 0$, and forcing function $f(t)$. We can rewrite this as a non-homogeneous differential equation

$$my'' + k_d y' + k_f y = f(t)$$

This is often called a forced vibration because there exists some forcing function $f(t)$. We can solve such initial value problems from using the Method of Undetermined Coefficients or Variation of Parameters for non-homogeneous second order linear differential equations.

Chapter 4

Systems of Differential Equations

4.1 Matrix Operations

As a reminder from any linear algebra course, a system

$$\mathbf{A}\vec{x} = \vec{B}$$

can be solved using Gaussian Elimination or Cramer's Rule methods. Let us also recall that the determinant of a matrix is

$$\det(\mathbf{A})$$

using expansion by minors, where across any row or column we find the alternating sums of determinant multiplied by its respective element. Furthermore, note that the following properties hold true.

1. $\begin{bmatrix} x \\ y \\ \vdots \end{bmatrix}' = \begin{bmatrix} x' \\ y' \\ \vdots \end{bmatrix}$
2. $(\mathbf{A}x)' = \mathbf{A}'x + Ax'$

Note that we will be trying to solve 2 equation systems in the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

4.2 Introduction to Systems

Let $\vec{Y} = \begin{bmatrix} y_1 \\ z_1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, and $\vec{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. So, we can create a homogeneous system with the equations

$$\vec{Y}' = \mathbf{A}\vec{Y}$$

and for a non-homogeneous system with the forcing function

$$\vec{Y}' = \mathbf{A}\vec{Y} + \vec{F}$$

Therefore, we will have homogeneous solutions

$$\vec{Y}_h = c_1 \begin{bmatrix} y_{1_h} \\ z_{1_h} \end{bmatrix} + c_2 \begin{bmatrix} y_{2_h} \\ z_{2_h} \end{bmatrix}$$

Furthermore, we can say that the solutions are **linearly independent** if and only if the determinant of the two solutions in a 2x2 matrix is not 0. A formal definition of linear independence states that there exists no c_1 and c_2 such that the sum of product of the solution and the constant will never be the zero vector.

We can express this with the Wronskian operation, as the Wronskian of two vector functions is

$$W(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}) = \det \left(\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix} \right)$$

With the Wronskian, we can say the two above vectors are linearly independent if the Wronskian is not equal to zero. So, for two homogeneous solutions, the Wronskian is either equal to or not equal to zero.

Definition 4.2.1: Solution Form

Given a homogeneous $(f_1, f_2 = 0)$ or non-homogeneous system of two differential equations

$$\begin{aligned} y' &= a_{1,1}y + a_{1,2}z + f_1 \\ z' &= a_{2,1}y + a_{2,2}z + f_2 \end{aligned}$$

which can also be expressed as

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} y \\ z \end{bmatrix} + \vec{F}$$

if $\begin{bmatrix} y_{1_h} \\ z_{1_h} \end{bmatrix}$ and $\begin{bmatrix} y_{2_h} \\ z_{2_h} \end{bmatrix}$ are two linearly independent homogeneous solutions and $\begin{bmatrix} y_p \\ z_p \end{bmatrix}$ is a solution of the non-homogeneous system, then the solution of the differential equation system can be expressed as

$$\begin{bmatrix} y_g \\ z_g \end{bmatrix} c_1 \begin{bmatrix} y_{1_h} \\ z_{1_h} \end{bmatrix} + c_2 \begin{bmatrix} y_{2_h} \\ z_{2_h} \end{bmatrix} + \begin{bmatrix} y_p \\ z_p \end{bmatrix}$$

However, we do not always start with a system of first-order differential equations. One option to solve second-order or higher-order equations is to convert them into a system of differential equations via a clever substitution. Given a second-order linear ODE in the form $ay'' + by' + cy = f$, we can make a substitution $z = y'$ such that we can now create a system

$$\begin{aligned} y' &= z \\ z' &= -\frac{c}{a}y - \frac{b}{a}z + \frac{1}{a}f \end{aligned}$$

4.3 Homogeneous Solutions

For some system $\vec{Y}' = \mathbf{A}\vec{Y}$, we can guess that the solutions y and z are in the form $y = pe^{rx}$ and $z = qe^{rx}$, respectively. If we plug these "guesses" into the system of equations, we arrive at the fact that the system now becomes

$$(\mathbf{A} - rI) \begin{bmatrix} p \\ q \end{bmatrix} = \vec{0}$$

which we can instantly recognize as the equation to find the eigenvalues r with identity matrix I. Note that this will provide us eigenvalues if and only if $\det(\mathbf{A} - rI) \neq 0$. Once we obtain our two eigenvalues r_1 and r_2 for a system of two first-order ODEs, we must find the corresponding eigenvectors to the eigenvalues with the homogeneous solutions expressed as

$$\vec{Y} = c_1 \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} e^{r_1 x} + c_2 \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} e^{r_2 x}$$

Let us begin with a simple example.

Example: System of First-Order ODEs

Suppose we have a system of differential equations be defined by

$$u' = u + 3v$$
$$v' = -2v$$

From this system, we can determine that this is a homogeneous system with matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$. To find the eigenvalues, we solve

$$\det \begin{bmatrix} 1-r & 3 \\ 0 & -2-r \end{bmatrix} = (1-r)(-2-r) = 0$$

Therefore, we find that $r_1 = 1$, $r_2 = -2$. To find the corresponding eigenvectors, we plug in each eigenvalue to find its corresponding eigenvector. For $r_1 = 1$, we plug this into the equation to find eigenvectors $(\mathbf{A} - rI)\vec{Y} = 0$ to get

$$\begin{bmatrix} 1-1 & 3 \\ 0 & -2-1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = 0$$

So, we get eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For our second eigenvalue $r_2 = -2$, we again plug it into a similar equation to get

$$\begin{bmatrix} 1+2 & 3 \\ 0 & -2+2 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = 0$$

which lets us get eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, our general solution to the system is

$$\vec{Y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2x}$$

Note that we only care about the *ratio* between p and q, as the ratios are uniquely determined by the eigenvalues and eigenvectors. Since the constants c_1 and c_2 are arbitrary, only the ratios between p and q matter. Furthermore, to obtain p and q, we must can also plug in the eigenvalues into the equation

$$(a_{1,1} - r)p + (a_{1,2})q$$

4.3.1 Double Roots

For repeated roots or eigenvalues, the solution would then be in the form

$$\vec{Y} = c_1 e^{rx} \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} + c_2 e^{rx} \left(\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} x + \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} \right)$$

Now, we must determine what the vectors $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$ and $\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ are. For our first vector, we can determine it simply as the eigenvector of eigenvalue r. For the second vector, however, the process is a little more complicated. We find a new linearly independent vector by solving the equation

$$(\mathbf{A} - rI) \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix}$$

which allows us to obtain a new vector $\begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$. Based on this new vector, the second solution can be expressed as $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} xe^{rx} + \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} e^{rx}$. Let us show an example for clarity.

Example: Repeated Eigenvalues

Solve the system defined by

$$y' = 3y - z$$
$$z' = 3z$$

First, we define $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$ such that we must solve the characteristic equation $(3 - r)^2 = 0$ where $r = 3$ is a repeated eigenvalue. Therefore, we can obtain our first eigenvector from the equation $(3 - 3)p + (-1)q = 0$, which shows us that $q = 0$ and we choose that $p = 1$. So, our first eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

To find our second solution, we must solve the system

$$\begin{bmatrix} 3-3 & -1 \\ 0 & 3-3 \end{bmatrix} \cdot \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which shows us that $q_2 = -1$ and we choose $p_2 = 0$. Therefore, our solution to the homogeneous system is

$$\vec{Y} = c_1 e^{3x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3x} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

4.3.2 Complex Eigenvalues

Lastly, we must discuss the case of complex eigenvalues. If we obtain the eigenvalues

$$r = a \pm bi$$

, we only need ONE of the two corresponding eigenvectors, in which we split off the real and imaginary parts to obtain our solution to the system. Therefore, our solution will be in the form

$$\vec{Y} = c_1(\text{Real Part}) + c_2(\text{Imaginary Part})$$

Let us show an example.

Example: Complex Eigenvalues

Suppose we have the system defined by

$$\begin{aligned} y' &= -y - z \\ z' &= 4y - z \end{aligned}$$

First, we find the eigenvalues from the characteristic equation $(-1-r)^2 + 4 = 0$ for which we obtain that the eigenvalues are $r = -1 \pm 2i$. Therefore, we can obtain our first eigenvector from the eigenvalue $-1+2i$ to be $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. Next, we separate the real and imaginary parts as follows.

$$\begin{aligned} \vec{Y} &= \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-1+2i}t \\ &= \begin{bmatrix} 1 \\ -2i \end{bmatrix} \left(e^{-t}\cos(2t) + e^{-t} i \sin(2t)\right) \\ &= \begin{bmatrix} e^{-t}\cos(2t) + ie^{-t}\sin(2t) \\ -2ie^{-t}\cos(2t) + 2e^{-t}\sin(2t) \end{bmatrix} \\ &= c_1 \begin{bmatrix} e^{-t}\cos(2t) \\ 2e^{-t}\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin(2t) \\ -2e^{-t}\cos(2t) \end{bmatrix} \end{aligned}$$

This is our solution. It can be easily shown that the two real and imaginary parts are indeed linearly independent.

4.4 Non-Homogeneous Systems

For non-homogeneous systems, we will try to find $\begin{bmatrix} y_p \\ z_p \end{bmatrix}$ by using the method of undetermined coefficients on both the y and z part. Let us demonstrate with an example. Note that our guesses for the particular solution should be analogous to that of second-order linear differential equations. However, if the two parts of the forcing function are different, the guess must be a sum of the individual guesses. So for example, if one of the forcing functions is $5t$ and the other is $\cos(t)$, then our guesses would be $At + B + C\cos(t) + D\sin(t)$ AND $Et + F + G\cos(t) + H\sin(t)$.

Example: Finding Particular Solution

Find the particular solution to the system of ODEs

$$\begin{aligned} y' &= -4y + 2z + 10 \\ z' &= -3y + 3z + 5t \end{aligned}$$

To find $\begin{bmatrix} y_p \\ z_p \end{bmatrix}$, we must make some guesses as to what y_p and z_p should look like. Let us guess that $y_p = At + B$ and $z_p = Ct + D$ for some constants A, B, C, D that we are to solve. If we plug these into the equation, we get

$$
\begin{aligned}
y' &= -4y + 2z + 10 \\
(4A - 2C)t + (A + 4B - 2D) &= 10
\end{aligned}
$$

and

$$
\begin{aligned}
z' &= -3y + 3z + 5t \\
C &= -3(At + B) + 3(Ct + D) + 5t \\
(3A - 3C)t + (C + 3B - 3D) &= 5t
\end{aligned}
$$

By matching constants of different orders of t, we get two systems

$$
\begin{aligned}
4A - 2C &= 0 \\
A + 4B - 2D &= 10 \\
3A - 3C &= 5 \\
C + 3B - 3D &= 0
\end{aligned}
$$

In any method you would like to solve this system (Gaussian Elimination or Cramer's Rule), we get that $A = -\frac{5}{3}$, $B = \frac{85}{18}$, $C = -\frac{10}{3}$, and $D = \frac{65}{18}$. Therefore, our particular solutions are

$$
\begin{bmatrix} y_p \\ z_p \end{bmatrix} = \begin{bmatrix} -\frac{5}{3}t + \frac{85}{18} \\ -\frac{10}{3}t + \frac{65}{18} \end{bmatrix}
$$

4.5 Variation of Parameters

The Variation of Parameters method is a more general way to find the particular solutions to a system of differential equations. We first assume that the particular solution is in some form

$$
\vec{Y_p} = Y_h \cdot \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
$$

where $Y_h = [Y_1, Y_2]$ such that the following where $F(t)$ are the forcing functions.

$$
Y_h \cdot \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = F(t)
$$

If we rearrange this equation, we get

$$
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}' = Y_h^{-1} F(t)
$$

which we can solve as where Y_h^{-1} is simply the inverse matrix of the solutions matrix.

$$
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \int Y_h^{-1} F(t)
$$

Recall from our initial assumption what the particular solution form is so, we can plug in what we have just obtained to get the solution as

$$\vec{Y_p} = Y_h \int Y_h^{-1} F(t) dt$$

Chapter 5

Laplace Transforms

Laplace Transforms are a way to "transform" differential equations into simple algebraic expressions. These are often used to handle discontinuous signal functions.

Definition 5.0.1: Laplace Transform

The Laplace Transform of $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Although Laplace Transforms are usually given in the form of a table, it is a good practice to try to derive some of the most common Laplace Transforms.

Example: Laplace Transform

Find the Laplace Transform of e^{at}.

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int +0^\infty e^{at} \cdot e^{-st}dt \\
&= \int_0^\infty e^{(a-s)t}dt \\
&= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty \\
&= 0 - \frac{1}{a-s} \\
&= \frac{1}{s-a}
\end{aligned}$$

A table of common Laplace Transforms is shown below.
Furthermore, the following properties of Laplace Transforms hold.

- The Laplace Transform holds the linearity property such that

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- The Laplace Transform exists if $f(t) \leq e^{Mt}$ for some $M \in \mathbb{R}$

$f(t)$	$F(s) = \mathfrak{L}\{f\}$	Domain of $F(s)$
1	$1/s$	$s > 0$
t^n (n positive integer)	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at}t^n, n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$s > \|b\|$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$s > \|b\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u(t-c)f(t-c)$	$e^{-cs}F(s)$	
$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	

5.1 Inverse Laplace Transforms

We start off with the definition of an Inverse Laplace Transform.

Definition 5.1.1: Inverse Laplace Transform

If $F(s) = \mathcal{L}\{f(t)\}$, then the **inverse Laplace Transform** is defined as

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Furthermore, the inverse Laplace Transform also holds the linearity property and can be found by trying to match the table with the function itself.

5.1.1 Solving IVPs

To solve IVPs using Laplace Transforms, we can take the Laplace Transform of both sides of the equation and simplify, then take the inverse Laplace Transform to reach our solution of the IVP. Let us show an example.

Example: Solving IVP

Solve $y' + y = e^{-3t}$ where $y(0) = 4$.
Using the Laplace Transform, we go through the following steps

$$\begin{aligned}
\mathcal{L}\{y' + y\} &= \mathcal{L}\{e^{-3t}\} \\
\mathcal{L}\{y'\} + \mathcal{L}\{y\} &= \mathcal{L}\left\{\frac{1}{s+3}\right\} \\
sY - y(0) + Y &= \mathcal{L}\{\frac{1}{s+3}\} \\
(s+1)Y &= \frac{1}{s+3} + 4 \\
Y &= \frac{1}{(s+3)(s+1)} + \frac{4}{s+1}
\end{aligned}$$

Now that we have found the Laplace Transform Y, we now want to reverse the Laplace Transform back to a friendlier form of $\mathcal{L}^{-1}\{Y\}$ to get the solution to the IVP.

$$\begin{aligned}
\mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s+1)} + \frac{4}{s+1}\right\} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{-0.5}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{0.5}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} \\
y(t) &= -0.5\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 4.5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
y(t) &= -0.5e^{-3t} + 4.5e^{-t}
\end{aligned}$$