AP Calculus BC

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Chapter 1

Introduction

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Calculus is the study of how things change. These notes cover Limits, Derivatives, Integrals, and Infinite Series, all of which are covered in the standard AP Calculus BC curriculums.

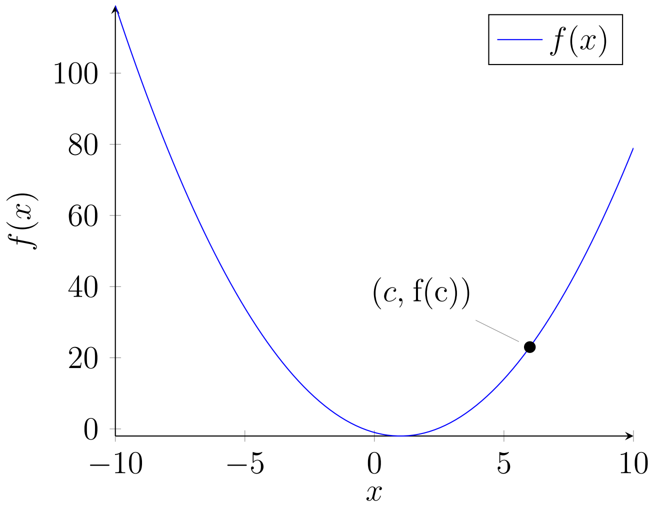
Chapter 2

Limits

2.1 Introduction to Limits

The number L is the *limit of a function* $f(x)$ as x approaches c if the function seems to approach c if the numbers x become extremely close to c. In mathematical terms, we can express this as

$$\lim_{x \to c} f(x) = c$$



Here, we can see that as we take values close to c in the function f(c), we will get values that approach f(c). This is what a limit is. If a function is not defined at f(c), the limit could still exist. A key takeaway is that limits do not equal the value of the function itself.

2.2 Right/Left-hand Limits

In order to determine if a limit exists, the left and right-hand limits must be compared.

The left-hand limit of f(x) can be written as

$$\lim_{x \to c^-} f(x)$$

This means the limit of f(x) as we take values approaching c but less than c.

The opposite, right-hand limits, can be written as

$$\lim_{x \to c^+} f(x)$$

This means we are finding the limit of f(x) as we take values approaching c but greater than c.

Definition 1. One/Two-Sided Limits

A limit is defined when the right hand limit is equal to the left hand side limit. This can be written as

$$\lim_{x \to c} f(x) = \lim_{x \to c^{-1}} f(x) = \lim_{x \to c^+} f(x)$$

When solving for limits, make sure to check both the right-hand limit and also the left-hand limit to make sure the limit exists. If the right hand and the left hand limits are not equal, the limit does not exist, or DNE for short.

Example 1

Given

$$f(x) = \begin{cases} y = 3x + 2, \ x < 2 \\ y = x^3, \ x \geq 2 \end{cases} \tag{2.1}$$

Find the $\lim_{x \to 2} f(x)$ using the piecewise function as defined above.

Let's take the left-hand limit first. When we approach $x = 2$ from the left-hand side, we follow the function $y = 3x + 2$ since x is less than 2. Thus, when we approach x = 2, we approach $3(2) + 2 = 8$.

When we take the right-hand limit, we approach $x = 2$ from the right-hand side, thus we follow the function $y = x^3$ when x is greater than 2. When we approach x = 2, we also approach $2^3 = 8$

Since both the left-hand and the right-hand limits are equal, the limit as x approaches 2 of f(x) is equal to 8.

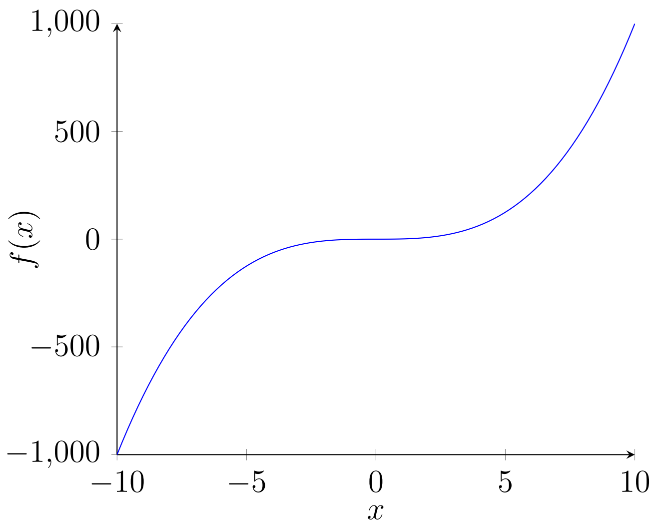
2.3 Infinite Limits & Limits at Infinity

We can also take limits as x approaches positive or negative infinity, as denoted by

$$\lim_{x \to \pm\infty} f(x) = L$$

Example 2

What would be $\lim_{x \to \infty} x^3$? Taking a look at the graph,



obviously the values would continue to get larger and larger and approach infinity. We call this limit **unbounded** and the $\lim_{x \to \infty} x^3$ would equal ∞.

If we were to find $\lim_{x \to -\infty} x^3$, the graph values of the function would continually become more negative, thus the limit would equal $-\infty$.

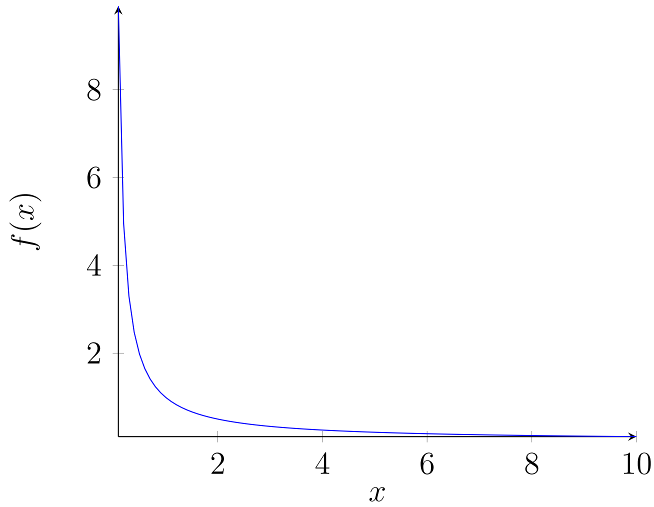
If instead a value approaches some real number when x approaches infinity, we call this value the horizontal asymptote of a function.

Definition 2. Horizontal Asymptote

If $\lim_{x \to \pm\infty} f(x) = L$ and L is a real number, we say that the line $y = L$ is the **horizontal asymptote** of f(x).

Example 3

The graph of $f(x) = \frac{1}{x}$ is shown below.



It looks like when x increases in value, the curve seems to become closer and closer to $y = 0$, but not actually touching this. This means that $\lim_{x\to\infty} \frac{1}{x} = 0$, and $y = 0$ is the horizontal asymptote of this function.

If x approaches some real number c and f(x) approaches $\pm\infty$, then we call this value the vertical asymptote of a function.

Definition 3. Vertical Asymptote

If $\lim_{x\to c} f(x) = \pm\infty$ and c is a real number, we say that the line $x = c$ is the **vertical asymptote** of f(x).

Example 4

Using the same graph of $f(x) = \frac{1}{x}$ above, It also looks like when x approaches 0, the function starts to increase dramatically and approach infinity. Thus, $\lim_{x\to 0} \frac{1}{x} = \infty$ and $x = 0$ is a vertical asymptote of this function.

Without using graphs, we can solve for these asymptotes algebraically using manipulations within the function.

2.4 Solving Limits

There are several methods to solving limits. Most involve algebraic manipulation to make the limit easier to find.

Method 1. Direct Substitution

When solving $\lim_{x \to c} f(x)$, plug c into f(x) to test if f(c) exists. If so, $f(c)$ would usually be the limit.

Method 2. Indeterminate Form

If direct substitution did not work and resulted in an indeterminate form, or $\frac{0}{0}$, try one of the following techniques:

1. Factoring - If the polynomials are factorable, cancel the terms and directly substitute in values
2. Conjugates - if there are radicals involved, multiply by its conjugate to remove the radical from the denominator and substitute directly
3. Trigonometric Identities - substitute in known identities to simplify the expressions

Method 3. Asymptotes

If direct substitution results in the form $\frac{b}{0}$, an asymptote was probably encountered. To identify a horizontal asymptote, divide the greatest power term on the top by the greatest power term on the bottom. If the result is a fraction, there is a horizontal asymptote at $y = \frac{a}{b}$. If it equals x raised to some positive power, then the graph approaches infinity or is unbounded. If it equals x raised to some negative power, the graph approaches 0.

To identify a vertical asymptote, factor and cancel out factors of the polynomial. The values of x that make the denominator equal to zero are the locations of the vertical asymptotes.

Method 4. Estimation

Estimate the limit by plugging in values close to $x = c$ into the function $f(x)$. If the function seems to approach some value, that may be the limit. If not, the function may not have a defined limit at that point. Be sure to use a table to organize the data.

Method 5. Graphical Approach

As a last resort, graph the function on a graphing calculator and observe the local behavior around the point. Take note of any discontinuities and asymptotes.

Take note of the following properties of limits. For $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and some constant k

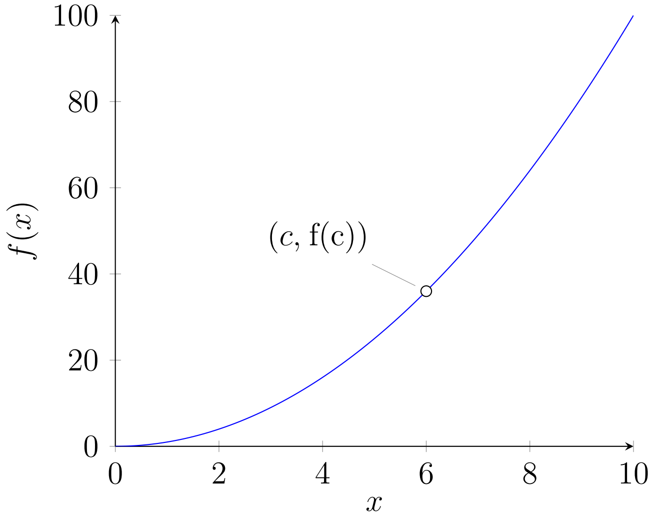
1. $\lim_{x\to c} f(x) \pm g(x) = L \pm M$
2. $\lim_{x\to c} f(x) \times g(x) = L \times M$
3. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$
4. $\lim_{x\to c} k \times f(x) = k \times L$
5. $\lim_{x\to c} f(x)^k = L^k$
6. $\lim_{x\to c} k = k$

2.5 Types of Discontinuities

There are three main types of discontinuities, as shown below.

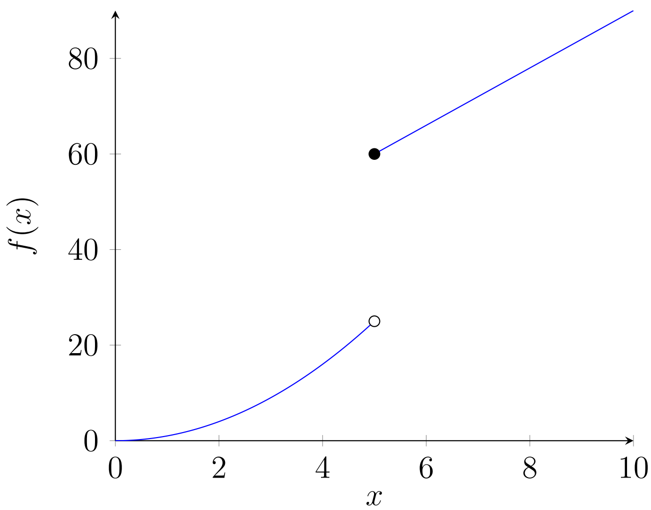
Definition 4. Removable Discontinuity

A "hole in the graph", or a point where the graph of a function is not connected but can be made connected by filling in a single point. An example of a removable discontinuity is shown below. There is a removable discontinuity at $(c, f(c))$.



Definition 5. Jump Discontinuity

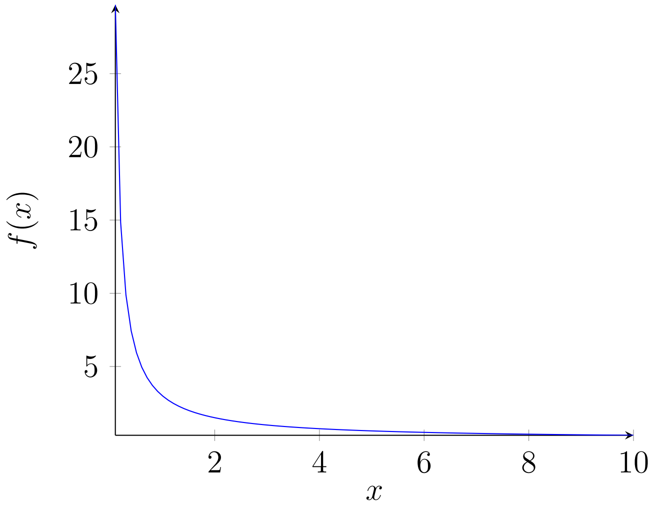
If the right and left-hand limits both disagree and do not approach infinity, then we have a jump discontinuity. This will look like a "jump" in the graph where a point will "jump" to another point when moving across two functions. An example of this is shown below.



There is a jump at $x = c$ where the function jumps to another value. Thus, there is no limit defined at this point.

Definition 6. Asymptotic Discontinuity

An asymptotic discontinuity is when values approach $\pm\infty$ and an asymptote is present. An example of this is shown below.



The function thus has two asymptotes: $y = 0$ and $x = 0$

2.6 Continuity

The opposite of discontinuities, or *continuity*, is a way of saying that a graph of a function is connected at a certain point or interval. We can say that a function or graph is continuous if it can be drawn without lifting up our pencil.

Definition 7. Continuity at a Point

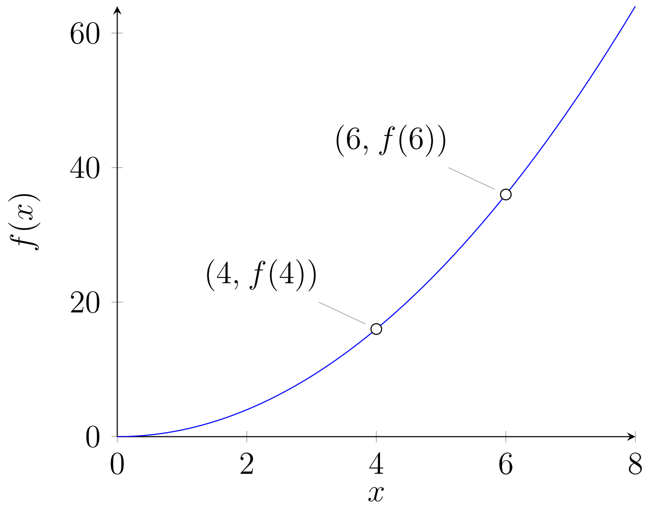
A function $f(x)$ is **continuous** at a point $x = a$ if
$\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a^-} = f(a)$ and $\lim_{x\to a^+} = f(a)$

Definition 8. Continuity over an Interval

A function $f(x)$ is **continuous** over interval $[a, b]$ if it is continuous at each point in the interval. This holds true if a and b are $-\infty$ and ∞ respectively.

Example 5

Given the graph of $f(x)$ below, determine if $f(x)$ is continuous over the interval $[4, 6]$



In order for a function to be continuous over the interval, it must be continuous at every point at that interval. Since the points $(4, f(4)$ and $6, f(6)$ are not defined, those points do not satisfy the conditions to be continuous. Thus, the function over the interval $[4, 6]$ is not continuous.

2.7 Theorems on Limits

There are two important theorems in this chapter on limits. They are shown below.

Theorem 1. Intermediate Value Theorem

Given any function f that is continuous over interval $[a, b]$, for any number L such that it is between values $f(a)$ and $f(b)$, there exists a number c located in the interval $[a, b]$ where

$$f(c) = L$$

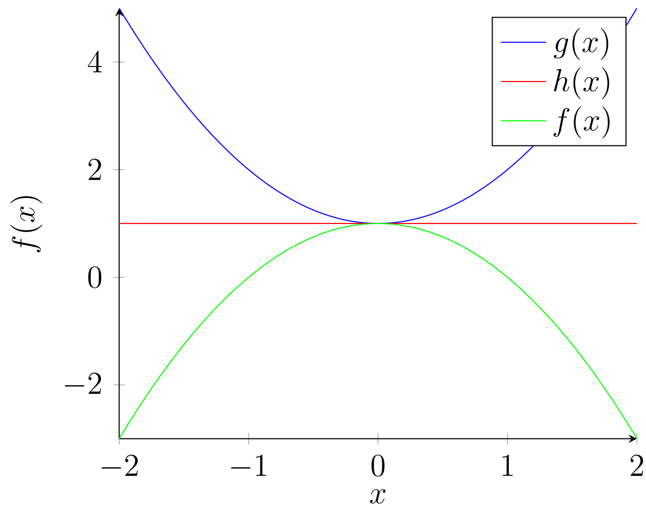
In simpler terms, this means that if a function is continuous over an interval $[a, b]$, the function takes on every value between $f(a)$ and $f(b)$. Pretty common sense right?

Theorem 2. Squeeze Theorem

Given three functions $g(x) \leq f(x) \leq h(x)$, and $\lim_{x \to a} g(x) = \lim x \to ah(x) = L$, then

$$\lim_{x \to a} f(x) = L$$

Since $f(x)$ must be between $g(x)$ and $h(x)$ and they are equal, $f(x)$ must equal that value as well. A visual of the Squeeze Theorem is shown below.



The functions $g(x)$ and $h(x)$ are equal at $x = 0$, where the inequality of $g(x) \leq f(x) \leq h(x)$ holds throughout the domain. Thus, the $f(x)$ must equal the other two functions when $g(x) = h(x)$. This theorem can be applied to many challenging limit problems, and choosing the right boundary functions is usually the hardest part.

2.8 Further Examples

Below are some examples on calculating limits.

Example 6

Find L for

$$L = \lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$$

$$\begin{aligned} L &= \lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4} \\ &= \lim_{x\to 2} \frac{(x-2)(x^2-2+4)}{(x+2)(x-2)} \\ &= \lim_{x\to 2} \frac{(x^2-2+4)}{(x+2)} \\ &= \lim_{x\to 2} \frac{((2)^2-2+4)}{((2)+2)} \\ &= \lim_{x\to 2} \frac{10}{4} = \boxed{3} \end{aligned}$$

Example 7

Find L for

$$L = \lim_{x\to\infty} \frac{2^{-x}}{2^x}$$

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{\frac{1}{2^x}}{2^x} \\ &= \lim_{x\to\infty} \frac{1}{2^x \times 2^x} \\ &= \lim_{x\to\infty} \frac{1}{4^x} \end{aligned}$$

Through direct substitution, we know that the function approaches 0 when x approaches infinity (since the denominator becomes larger and larger). Thus, $\boxed{L = 0}$

Example 8

Find L for

$$L = \lim_{h \to 0} \frac{(4+h)^2 - 16}{h}$$

$$\begin{aligned} L &= \lim_{h \to 0} \frac{16 + 8h + h^2 - 16}{h} \\ &= \lim_{h \to 0} \frac{h(h+8)}{h} \\ &= \lim_{h \to 0} h + 8 \end{aligned}$$

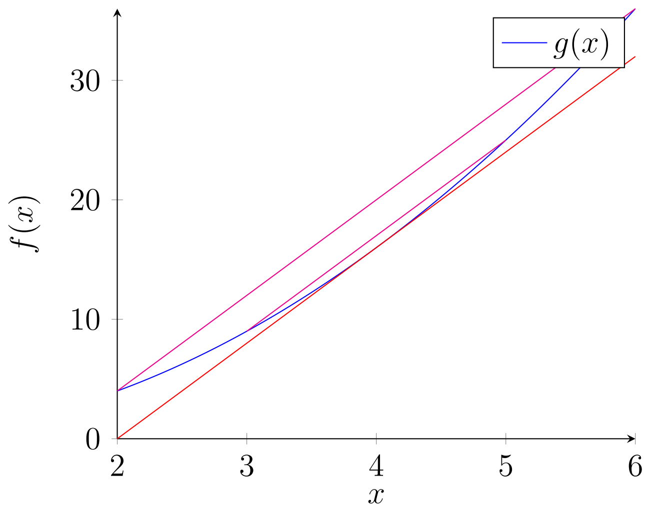
Through direct substitution, we see that the limit is equal to $0 + 8 = \boxed{8}$

Chapter 3

Derivatives

3.1 Introduction to Derivatives

To put it simply, a derivative is the slope of a graph at a point. This could also mean the slope of the *tangent line* at a point, or the *instantaneous rate of change*. A tangent line can be approximated with a secant line, where the two endpoints come closer and closer together until they meet. A visual can be found below.



The red line is the tangent line, and the magenta lines connecting two points are called secant lines. The slope of the red line (the tangent line), is the derivative at the point of tangency.

The formula for slope is

$$y = \frac{y_2 - y_1}{x_2 - x_1}$$

for points (x_2, y_2) and (x_1, y_1). When we move these points together, the points should almost coincide, and we can take the limit of this slope formula to find the definition of a derivative.

A derivative of a function $y = f(x)$ can be denoted as $\frac{dy}{dx}$, $\frac{df}{dx}$, or $f'(x)$. The first two notations are called Leibniz notations and the last notation is called Lagrange notation.

Definition 9. Derivative

The derivative of a function $f(x)$ at $x = a$ can be defined as

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The general derivative function of $f(x)$ is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For the second definition, Δx as h may be seen in some textbooks.

The first definition of the derivative is finding the derivative, or slope of the tangent line, at a given point $x = a$. The second definition gives the general derivative function that represents the slope at every point.

A derivative can be interpreted as an instantaneous rate of change or the slope of a tangent line. For example, the rate of change of position would be the velocity, and the rate of change of velocity is acceleration.

If we find the derivative of the derivative of a function, we call this the second-derivative of a function, denoted by $\frac{d^2 f}{dx^2}$ or $f''(x)$. Higher level derivatives can be written using similar notations. The second-derivative can be thought of as the rate of change of the rate of change, or the rate of change of the slope, or acceleration for a position function.

3.2 Solving Derivatives

The formal definition of a derivative does not provide an easy method for finding general derivatives. Below is a list of formulas that will be used constantly throughout Calculus. Assume x is some variable and k is some constant.

Addition/Subtraction	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
Constant	$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x) \times g(x)] = f'(x)g(x) + g'(x)f(x)$
Quotient Rule	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Example 9

Find the derivative of $f(x) = x^2$. Then calculate the derivative at $x = 2$.

Using power rule, $\frac{d}{dx}[x^2] = 2x^{2-1} = \boxed{2x}$

At the point $x = 2$, the derivative would be $2(2) = \boxed{4}$

We can interpret this as the slope of the tangent line of $f(x) = x^2$ is equal to 4, or the instantaneous rate of change here is 4. The derivative function, $f'(x) = 2x$, gives the derivative at any point x.

Assume that x is some variable, and the a and k are arbitrary constants. Then, these derivative identities should hold true. These identities should be memorized.

$\frac{d}{dx}[k] = 0$	$\frac{d}{dx}[x] = 1$	$\frac{d}{dx}[a^x] = a^x \ln(a)$
$\frac{d}{dx}[e^x] = e^x$	$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\frac{d}{dx}[\cos(x)] = -\sin(x)$
$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$	$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\frac{d}{dx}[\csc(x) = -\csc(x)\cot(x)$	$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln(a)}$
$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$

Example 10

Find the derivative of $y = \sqrt{x^2 + 2x - 1}$

First off, recognize that this is a composition of functions, where the outer function ($\sqrt{x}$) is taking in the function $(x^2 + 2x - 1)$ as its input. Thus, we can use the chain rule which states that $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{df}{dg} \times \frac{dg}{dx}$.

Using this, we can go through the following steps.

$$\begin{aligned}
\frac{d}{dx}[\sqrt{x^2 + 2x - 1}] &= \frac{d}{dx}(x^2 + 2x - 1)^{\frac{1}{2}} \\
&= \frac{d}{d(x^2 + 2x - 1)}(x^2 + 2x - 1)^{\frac{1}{2}} \times \frac{d}{dx}[x^2 + 2x - 1] \\
&= \frac{1}{2}(x^2 + 2x - 1)^{\frac{1}{2}-1}[2x + 2] \\
&= \boxed{\frac{x + 1}{\sqrt{x^2 + 2x - 1}}}
\end{aligned}$$

Most derivative problems will involve some sort of chain rule, which can be quite tricky to implement without practice. Below is a problem set that may be helpful.

https://tutorial.math.lamar.edu/problems/calci/chainrule.aspx

Example 11

Calculate the derivative of $y = \sin(2x)\cos(2x)$ at $x = \frac{\pi}{2}$

There is a product of two functions as well as composition of functions for the sine and cosine. We can first apply product rule, then chain rule when finding the derivative of specific functions.

$$\begin{aligned}\frac{d}{dx}[\sin(2x)\cos(2x)] &= \frac{d}{dx}[\sin(2x)]\cos(2x) + \frac{d}{dx}[\cos(2x)]\sin(2x) \\ &= \cos(2x)\frac{d}{dx}[2x]\cos(2x) + (-\sin(2x))\frac{d}{dx}[2x]\sin(2x) \\ &= 2\cos^2(2x) - 2\sin^2(2x)\end{aligned}$$

Evaluating at $x = \frac{\pi}{2}$, we have $2\cos^2(2(\frac{\pi}{2})) - 2\sin^2(2(\frac{\pi}{2})) = \boxed{2}$

Example 12

Find the derivative of $\sec a^x$

We first want to apply chain rule to find the derivative of the outer function $\sec x$, then multiply that by the derivative of the inside function a^x to find our derivative. This is a direct application of chain rule.

$$\begin{aligned}\frac{d}{dx}[\sec(a^x)] &= \frac{d}{d(a^x)}[\sec a^x \frac{d}{dx}[a^x] \\ &= \boxed{\sec(a^x)\tan(a^x)(a^x\ln(a))}\end{aligned}$$

3.3 Implicit Differentiation

When we want to differentiate an implicit function, or one where y is not explicitly defined as a function of x, we can use the chain rule to help us differentiate and find $\frac{dy}{dx}$. When we encounter x's, we will differentiate them normally using standard rules, but when we encounter y's, we will treat them as functions of x, or $y(x)$. Thus, we will use chain rule and differentiate with respect to y then multiplied by $\frac{d}{dx}[y] = \frac{dy}{dx}$ since the derivative of the inner function y, is $\frac{dy}{dx}$.

Example 13

Find the $\frac{dy}{dx}$ if $x^2 + y^2 = 4$

Using implicit differentiation, we want to differentiate this function normally, but treat y as a function of x. Thus, the steps to differentiate would be:

$$\begin{aligned}
\frac{d}{dx}[x^2 + y^2] &= \frac{d}{dx}[4] \\
\frac{d}{dx}[x^2] + \frac{d}{dx}[y(x)^2] &= 0 \\
2x + \frac{d}{d(y)}[y(x)^2]\frac{d}{dx}[y] &= 0 \\
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \boxed{\frac{-x}{y}}
\end{aligned}$$

We differentiated x normally, and y as a function of x using chain rule.

Example 14

Find $\frac{dy}{dx}$ if $x - 2y = 2x$

First, we should group all of the terms on one side of the equation then differentiate implicitly like the answer above.

$$\begin{aligned}
\frac{d}{dx}[-x - 2y] &= \frac{d}{dx}[0] \\
\frac{d}{dx}[-x] - \frac{d}{dx}[2y(x)] &= 0 \\
-1 - 2\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \boxed{-\frac{1}{2}}
\end{aligned}$$

Below is a trick for implicit differentiation using partial derivatives, a concept in Calculus

3.

Definition 10. Implicit Differentiation Trick

Given function $F(x, y) = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Where F_x is the derivative of $F(x, y)$ when y is treated as a constant, and F_y is the derivative of $F(x, y)$ when x is treated as a constant. These are also called **partial derivatives**.

Using the above example, we have

$$F_x = 2x \text{ and } F_y = 2y, \text{ when opposite variables are treated as constants, so}$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

This answer is the same answer as before, verify to see this is true. Disclaimer: Do not use this trick formula on any tests or the AP exam where work must be shown, since teachers may not recognize this formula as a viable formula.

3.4 Derivatives of Inverse Functions

The derivative of an inverse function can be derived from the chain rule.

Definition 11. Inverse Function

A function $f(x)$ is the inverse of function $g(x)$ if it is the reflection of $g(x)$ over the line $y = x$. In other words
$$f(g(x)) = x$$

If we differentiate the function above, $f(g(x)) = x$ on both sides, we get by using chain rule

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$$
$$f'(g(x))g'(x) = 1$$
$$g'(x) = \frac{1}{f'(g(x))}$$

Definition 12. Derivative of Inverse Function

Given function $f(x)$ and inverse function $g(x)$,

$$g'(x) = \frac{1}{f'(g(x))}$$

Example 15

Given $g(2) = 2$ and $f'(x) = x^2$, find $g'(2)$

Using the formula for derivatives of inverse functions, we get

$$g'(2) = \frac{1}{f'(2)} = \frac{1}{2^2} = \boxed{\frac{1}{4}}$$

Chapter 4

Applications of Derivatives

4.1 Position, Velocity, and Acceleration

When given a position function locating a particle, the **rate of change** or how fast something is going at each position is the velocity of the particle. When given a function representing the velocities of a particle with respect to time, the rate of change of the velocity will give us the acceleration of the particle. These operations of rates of changes can be used with derivatives as well. The below chart demonstrates how derivatives connect position, velocity, and acceleration functions.

$$\text{Position} \overset{\frac{d}{dx}}{\Longrightarrow} \text{Velocity} \overset{\frac{d}{dx}}{\Longrightarrow} \text{Acceleration}$$

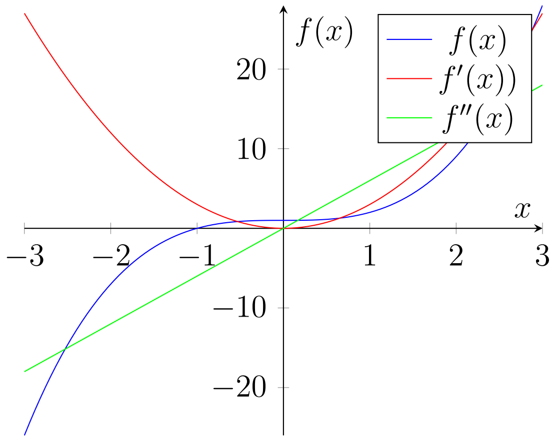
When interpreting velocity, a positive velocity means that a particle is moving forwards and a negative velocity means the particle is moving backwards. If the velocity is zero, then the particle is not moving. Speed, however, is the absolute value of velocity (since it is always positive).

In addition, the rate of change of the velocity is the acceleration, but the rate of change of speed depends on the direction of movement. The chart below summarizes the relationships.

Velocity	Acceleration	Speed
Positive	Positive	Increasing
Negative	Positive	Decreasing
Positive	Negative	Decreasing
Negative	Negative	Increasing

4.2 Graphing Position, Velocity, and Acceleration

Velocity is the derivative of the position function, and acceleration is the derivative of velocity. Below is a visualization of some example functions and their respective derivatives. $f(x)$ is a position graph, $f'(x)$ is analogous to a velocity graph, and $f''(x)$ is an acceleration graph.



Looking at the blue curve, $f(x)$, we see that the slope starts off very steep (positive), becomes less and less steep until it hits $x = 0$ where the slope is 0, then the slope increases again. Thus, $f'(x)$ follows the values of the slope, starting very positive, then going to 0, then increasing a lot as $f(x)$ increases.

$f''(x)$, however, is the slope of $f'(x)$. Looking at $f'(x)$, the slope starts very negatively, then approaches 0 at $x = 0$, then increases and becomes very positive. Thus, $f''(x)$ follows these values.

4.3 Related Rates

Related rate problems are problems that involve solving for the rate of increase or decrease of a particular value in a word problem. Examples of related rate problems may include finding the rate of change of a length between two objects, the volume of a solid, or the angle between two objects.

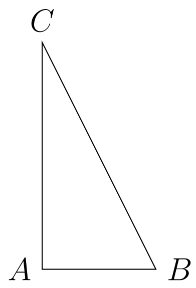
Method 6. Solving Related Rates Problems

When solving related rates, follow the below guidelines and steps:

1. List out all formulas (ex. Area, Volume) related to the problem statement
2. Implicitly differentiate the formula, generally with respect to t or time
3. Solve for any unknowns using the given values (not the derivatives/rates). This may require proportions to be used which are given at the beginning of the problem.
4. Plug in all values and given rates into the implicitly differentiated formula, then solve for the rate asked to find.

Example 16

If one leg of AB of a right triangle increases at a rate of $\frac{1}{6}$ feet per second, while the other leg AC decreases at $\frac{1}{4}$ inches per second, find how fast the hypotenuse is changing when $AB = 6$ feet and $AC = 8$ feet.



We know that $\frac{dAB}{dt} = \frac{1}{6}$ and $\frac{dAC}{dt} = -\frac{1}{4}$. Since at any time, $AB^2 + AC^2 = CB^2$, then by implicit differentiation,

$$2AB\frac{dAB}{dt} + 2AC\frac{dAC}{dt} = 2CB\frac{dCB}{dt}$$

At the instant where $AB = 6$ and $AC = 8$, $CB = 10$ by Pythagorean Theorem. Plugging in all of the rates and values from above, we have

$$2(6)(\frac{1}{6}) - 2(8)(\frac{1}{4}) = 2(10)\frac{dCB}{dt}$$

Since we are solving for the rate at which the hypotenuse is changing, or $\frac{dCB}{dt}$, we solve for that rate. $\frac{dCB}{dt} = \boxed{-\frac{1}{10}}$

4.4 Local Linearization

Differentiable graphs, meaning the derivative exists along the curve, have a property called **local linearity**, where the tangent line can be used to approximate values close to the curve. The method below lists the steps to find a relatively close point to the original point using the local linearity property.

Method 7. Local Linearization

Given function $f(x)$ or derivative $f'(x)$ and a point along $f(x)$, or (x_1, y_1)

1. Find the derivative at $x = a$, or $f'(a)$
2. Write the tangent line function in point slope form at $x = a$. $y - y_1 = f'(a)(x - x_1)$
3. Plug in the approximated value, $x = c$, into the tangent line function

Example 17

Estimate $\sqrt{4.36}$ using the local linearity property with function $f(x) = \sqrt{x}$ at $x = 4$.

First of all, we want to find the derivative of $f(x)$ for the tangent line function.

$$\begin{aligned}\frac{d}{dx}[\sqrt{x}] &= \frac{d}{dx}[x^{\frac{1}{2}}] \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

Plugging $x = 4$, we get $f'(4) = \frac{1}{4}$. At $x = 4$, $f(x) = 2$, so we will be using point-slope form at point $(4, 2)$. With the formula for point-slope form, we get for the tangent line function

$$y - 2 = \frac{1}{4}(x - 4)$$

Now we plug in 4.36 into this function to approximate.

$$y - 2 = \frac{1}{4}(4.36 - 4)$$

Solving this equation we get $y \approx \boxed{2.09}$

4.5 L'Hopital's Rule

L'Hopital's rule is a useful method for finding limits when direct substitution results in an indeterminate form: $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $\frac{-\infty}{-\infty}$.

Definition 13. L'Hopital's Rule

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

Example 18

Find $\lim_{x\to 0} \frac{\sin(x)}{x}$

We always want to first try direct substitution, but since it results in an indeterminate form $\frac{0}{0}$, we can now use L'Hopital's rule. By L'Hopital's rule,

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[x]} \\ &= \lim_{x\to 0} \frac{\cos(x)}{1} \\ &= \lim_{x\to 0} \cos(x)\end{aligned}$$

By direct substitution on the last equation, we get $\cos(0) = \boxed{1}$

L'Hopital's rule may need to be applied several times before it results in a form that isn't indeterminate. Use L'Hopital's generally as a last resort when resulting in an indeterminate form, since differentiating the rational functions can get messy.

4.6 Theorems on Derivatives

Theorem 3. Differentiability Implies Continuity

If a function is differentiable at point $x = a$, then it is continuous at point $x = a$.

If a function is differentiable over interval $[a, b]$, then it is also continuous over interval $[a, b]$.

The converse of the above theorem does not necessarily hold true.

Theorem 4. Mean Value Theorem

If $f(x)$ is continuous over $[a, b]$ and differentiable over (a, b), then there is some number c, such that $a < c < b$, and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, it states that if $f(x)$ is differentiable over the interval (a, b), then there exists a point where the instantaneous rate of change (slope of tangent line) equals the average rate of change (slope of secant line).

Theorem 5. Rolle's Theorem

If $f(x)$ is continuous over interval $[a, b]$, differentiable over interval (a, b), and $f(a) = f(b)$, then there is a number c such that $a < c < b$ and

$$f'(c) = 0$$

This theorem is a special case of the Mean Value Theorem, where the average rate of change is 0 because the endpoints have equal values.

Theorem 6. Extreme Value Theorem

If $f(x)$ is continuous over interval $[a, b]$, then there exists

1. An absolute maximum value of $f(x)$ over the interval $[a, b]$
2. An absolute minimum value of $f(x)$ over the interval $[a, b]$

The Extreme Value Theorem is trivial, but the conditions of the theorems should be familiarized.

4.7 Critical Points, Maxima and Minima

Definition 14. Critical Point

A critical point is a point $(x, f(x))$ on a graph of a function where either

1. $f'(x) = 0$
2. $f'(x)$ is undefined

Definition 15. Relative Maximum/Minimum

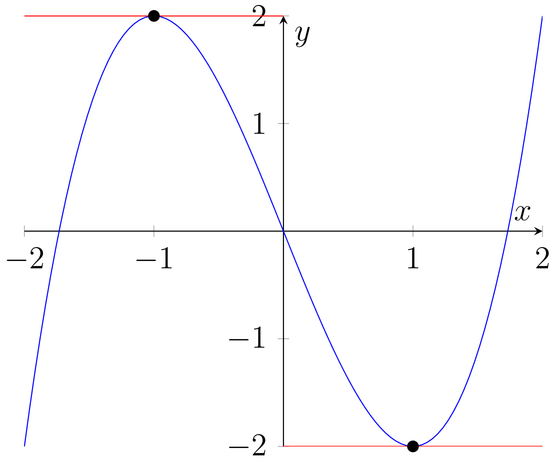
A relative maximum point is a point where the function changes from increasing to decreasing, and the values near the point are all less than the point.

A relative minimum point is a point where the function changes from decreasing to increasing, and the values near the point are all greater than the point.

A key takeaway is that all relative extremas (maxima/minima) are critical points.

Example 19. Recognizing Critical Values and Relative Maxima/Minima

Locate all critical points and relative maxima and minima on the function $f(x) = x^3 - 3x$ graphed below.



At points $(-1, 2)$ and $(1, -2)$, it is pretty obvious that the slope at each point is 0. Thus, these two points are critical values in the graph shown. There are no places where the derivative is undefined , so those two points are our only two critical points.

Taking a look at $(-1, 2)$, it seems as if it is greater than all of the values around it, so we call it a **relative maximum**. Notice how the derivative or slope is increasing then decreasing around it.

For the points $(1, -2)$, it is pretty obvious that this is a **relative minimum**, since it is a dip in the graph and the derivative goes from decreasing to increasing.

Now if we take a look at this algebraically, we want to find the derivative of the function $f(x) = x^3 - 3x$ and set it equal to 0 to find critical values.

$$\begin{aligned}\frac{d}{dx}\left[x^3 - 3x\right] &= \frac{d}{dx}\left[x^3\right] - \frac{d}{dx}[3x] \\ &= 3x^{3-1} - 3x^{1-1} \\ &= 3x^2 - 3\end{aligned}$$

Since the derivative $f'(x) = 3x^2 - 3$ is defined for all reals, we only have to find where it equals 0 for critical values.

$$\begin{aligned}3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ (x+1)(x-1) &= 0 \\ x &= \pm 1\end{aligned}$$

Therefore, we have critical values at points $x = -1$ and $x = 1$. Now analyzing the behavior around these points. For all $f'(x) = 3x^2 - 3$ less than $x = -1$, $f'(x)$ is positive. For all $f'(x) = 3x^2 - 3$ between $x = -1$ and $x = 1$, $f'(x)$ is negative. For all $f'(x)$ greater than $x = 1$, $f'(x)$ is positive.

Since at $x = -1$, $f'(x)$ goes from **positive to negative**, or $f(x)$ goes from **increasing to decreasing**, the value of $f(x)$ at $x = -1$ is a **relative maximum**.

Also at $x = 1$, $f'(x)$ goes from **negative to positive**, for $f(x)$ goes from **decreasing to increasing**, the value of $f(x)$ at $x = 1$ is a **relative minimum**.

Refer to the graph above to visualize the above statements.

When finding relative extrema, draw out a number line with the critical values and test whether the derivative in points in each interval is positive or negative. Conclusions can be drawn from the derivatives between critical values.

Definition 16. Absolute Extrema

An **absolute maxima** of a function $f(x)$ over interval $[a, b]$ is the value where the function attains the greatest value.

An **absolute minima** of a function $f(x)$ over interval $[a, b]$ is the value where the function attains the smallest value.

Method 8. Finding Absolute Extrema

1. Find all critical points
2. Find whether each critical point is a relative minima, maxima, or neither
3. Test all **relative maxima and minima and endpoints of the interval** into the function, the smallest and greatest value will be the absolute extrema of $f(x)$ over the given interval

4.8 Increasing and Decreasing Intervals

Sometimes, questions or problems will ask to find the increasing or decreasing intervals of a given function. The process is pretty intuitive, which is shown below.

Method 9. Finding Increasing Decreasing Intervals

To find the increasing or decreasing intervals of a function:

1. Find all critical points
2. Order the critical points on a number line
3. Test each space between two critical points with the derivative function to see if it is positive or negative
4. The positive derivative indicates that it is an increasing interval, and a negative derivative corresponds to a decreasing interval

Example 20

Find the increasing and decreasing intervals of $f(x) = x^3 - 3x$

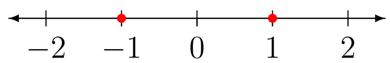
First we want to find the derivative to solve for the critical points.

$$\begin{aligned} \frac{d}{dx}\left[x^3 - 3x\right] &= \frac{d}{dx}\left[x^3\right] - \frac{d}{dx}\left[3x\right] \\ &= 3x^2 - 3 \end{aligned}$$

With our derivative function $f'(x) = 3x^2 - 3$, we can now find the critical points when the function is set equal to 0.

$$\begin{aligned} 3x^2 - 3 &= 0 \\ x &= \pm 1 \end{aligned}$$

With our critical points, we now list them out on a number line.



We will need to test the intervals:

1. $(-\infty, -1)$
2. $(-1, 1)$
3. $(1, \infty)$

By trying values in these regions, we find that from the interval $(-\infty, -1)$, the interval is increasing because the derivative is positive, and from the interval $(-1, 1)$, the derivative is negative so the interval is decreasing, and the last interval is increasing.

4.9 Concavity and Inflection Points

Concavity is a mathematical way of determining whether the rate at which a function is changing is increasing or decreasing. They can be found with second derivatives of functions.

Definition 17. Concavity

A function $f(x)$ is **Concave Up** over an interval I if all tangents to the curve $f(x)$ over interval I are below the graph of $f(x)$, or $f''(x) > 0$

A function $f(x)$ is **Concave Down** over an interval I if all tangents to the curve $f(x)$ over interval I are above the graph of $f(x)$, or $f''(x) < 0$

To find intervals of concavity, calculate the second derivative of the function and set it to either >0 or <0 depending on the type of concavity listed above. The method of intervals on a number line from the previous section may help.

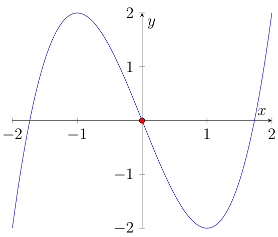
Definition 18. Inflection Point

A point on a continuous plane curve at which the curve changes from being concave down to concave up, or concave up to concave down. In other words, this is when $f''(x)$ changes signs.

Method 10. Finding Inflection Points

1. Find the second derivative
2. Find points where $f''(x)$ equals 0 or undefined
3. If there is a sign change for the second derivative at one of the points, it is an inflection point

Example 21



Given this graph, we can obviously tell the intervals where the function is concave down and concave up. If it is concave down, it will look like an upside down **U**, where if it is concave up, it will look like a **U**. The inflection point is at the origin because this is where it changes from concave down to concave up.

4.10 Second Derivative Test

The second derivative test is another way to determine the relative extrema using the second derivative.

Method 11

Second Derivative Test Given critical point where $f'(c) = 0$, if

1. $f''(c) < 0$, then $f(x)$ has a relative maximum at c
2. $f''(c) > 0$, then $f(x)$ has a relative minimum at c
3. $f''(c) = 0$, inconclusive

Example 22

Find the relative extrema of $y = x^3 - 3x + 4$ using the second derivative test.

First, we need to find the critical points of the function.

$$f'(x) = 0 = 3x^2 - 3$$

So the critical points would be at $x = \pm 1$. Finding the second derivative for the second-derivative test, we get

$$f''(x) = 6x$$

At $x = 1$, since $f''(x) > 0$, it is a relative minimum.

At $x = -1$, since $f''(x) < 0$, it is a relative maximum.

4.11 Optimization

Optimization is when we are given a function, and we are asked to find a maximized or minimized value. We can use derivatives to help us, as well as some of the methods from the previous sections.

Method 12. Steps to Solve Optimization Problems

When given an optimization problem, the following steps can provide some helpful guidelines:

1. Introduce all variables
2. Determine which quantity is to be optimized, and over what interval
3. Identify any formulas relating quantities in terms of the variables
4. Write any equations relating the variables
5. Locate the absolute minima and maxima

Optimization problems are probably one of the most difficult problems to solve, but they generally have a pattern to them. They usually involve areas, volumes, and distances.

Example 23

Let x and y be two nonnegative numbers such that $x + 2y = 50$ and $(x + 1)(y + 2)$ is maximized. Find x and y

Since we are given the constraint $x+2y = 50$, and the function $f = (x+1)(y+2)$ that is to be maximized, we can work with these two to simplify equations by substitution.

Solving for x in the constraint, we get $x = 50 - 2y$, for which we can substitute into our optimized function.

$$\begin{aligned} f &= (x + 1)(y + 2) \\ &= (50 - 2y + 1)(y + 2) \\ &= (51 - 2y)(y + 2) \\ &= 102 + 47y - 2y^2 \end{aligned}$$

With this function to optimize, we can go ahead and find the maximum by finding critical points and testing the local behaviors around them.

$$\begin{aligned} \frac{d}{dy}\left[f(y)\right] &= \frac{d}{dy}\left[102 + 47y - 2y^2\right] \\ &= 47 - 4y \end{aligned}$$

Setting $47 - 4y = 0$, we get $y = \frac{47}{4}$, testing the local behavior with points near it; it indeed is a maximum since the function goes from increasing to decreasing. We know this is the absolute maximum since this is the vertex of a downward facing parabola.

Thus, $\boxed{y = \frac{47}{4} \text{and} x = \frac{53}{2}}$

Example 24

What are the dimensions of a top-less cylinder that would minimize the surface area while having a volume of 30cm^3?

First, we want to find all equations relevant to the circumstances. We have:

1. $\pi r^2 h = 30$, for the volume
2. $A = 2\pi rh + \pi r^2$, for the surface area (remember, it is a topless cylinder)

Let's first solve for h in the volume equation.

$$h = \frac{30}{\pi r^2}$$

Plugging this into the surface area function gives,

$$A(r) = 2\pi r(\frac{30}{\pi r^2}) + \pi r^2 = \frac{60}{r} + \pi r^2$$

Now we want to minimize this function for the surface area, so we must find the critical point(s) and make sure it is indeed a minimum.

$$A'(r) = -\frac{60}{r^2} + 2\pi r = \frac{2\pi r^3 - 60}{r^2}$$

Solving for it where $A'(r) = 0$, the only critical point is $r = \sqrt[3]{\frac{60}{2\pi}} \approx 2.1216$

The second derivative of this volume function is,

$$A''(r) = \frac{120}{r^3} + 2\pi$$

Evaluating this at the critical point, we see that it is indeed positive, showing that $A(r)$ is always concave up and that the critical point found is indeed an **absolute minimum**.

Plugging $r = 2.1216$ into our original equation for height in terms of radius, we get

$$\frac{30}{\pi(2.1216)^2} \approx 2.1215$$

Thus, the final dimensions are $\boxed{\text{r} = 2.1216}$ and $\boxed{\text{h} = 2.1215}$

Example 25

Find the point on the curve $y = x - 3$ that is closest to the origin.

First, we want to set up a function that would give us the distance from a point on the function to the origin, to which we can optimize.

The function $f(x) = \sqrt{(x-0)^2 + (y+0)} = \sqrt{x^2 + y^2}$ would give us the distance from the origin using the distance formula. Plugging in the function $y = x - 3$ for y, we get

$$D(x) = \sqrt{x^2 + (x-3)^2} = \sqrt{x^2 + x^2 - 6x + 9} = \sqrt{2x^2 - 6x + 9}$$

To optimize this distance function, we find the critical points then test whether or not it is a minimum using the second derivative test. Thus,

$$D'(x) = \frac{4x - 6}{2\sqrt{2x^2 - 6x + 9}}$$

The critical points would be $x = \frac{3}{2}$ (since the denominator's roots aren't real), and testing the local behavior, it changes from decreasing to increasing, so $x = \frac{3}{2}$ is the point where it is closest to the origin.

Chapter 5

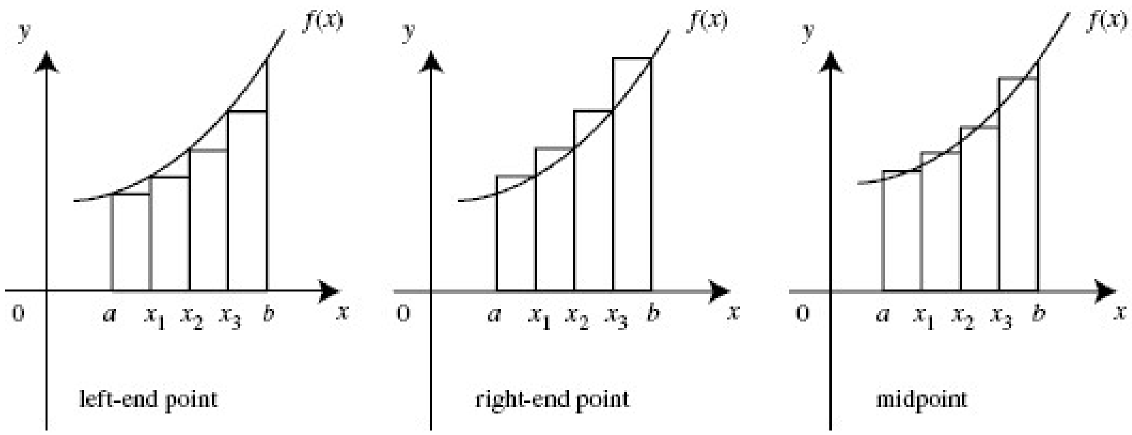
Integration

5.1 Riemann Sums

Riemann Sums is a technique used to approximate the area under a curve over a defined interval using rectangles or trapezoids. Using the values given from a function, we can construct many different types of rectangles and trapezoids, each of which giving us a different approximation. Each approximation may be less or more accurate, depending on the concavity of the function over the interval.

Rectangular riemann sums can be represented in three forms:

1. **Right Riemann Sum** - this means that the rectangles touch the curve at the upper right vertex
2. **Left Riemann Sum** - this means that the rectangles touch the curve at the upper left vertex
3. **Midpoint Sum** - this means that the rectangle intersects the curve at the middle of the upper side



Example 26

Find the area under the curve $f(x) = x^2 + 3$ with rectangles in a left riemann sum of width 1 from $x = 0$ to $x = 4$

The lengths of the rectangles are the following: $f(0) = 3$, $f(1) = 4$, $f(2) = 7$, $f(3) = 12$

**$f(4)$ is not included because it is not used in a left riemann sum

Thus, the sum of the areas of the rectangles each with width 1 is $3 \times 1 + 4 \times 1 + 7 \times 1 + 12 \times 1 = \boxed{26}$

Definition 19. Estimations with Riemann Sums

Right Riemann Sums:
If a curve is increasing over an interval, then the right riemann sum over the interval is an **overestimate** of the area under the curve. If the curve is decreasing over the interval, then the right riemann sum is an **underestimate** of the area under the curve.

Left Riemann Sums: If a curve is increasing over an interval, then the right riemann sum over the interval is an **underestimate** of the area under the curve. If the curve is decreasing over the interval, then the right riemann sum is an **overestimate** of the area under the curve.

Midpoint Sums: The midpoint sum estimation depends on the concavity of the curve, which will not be covered in these notes.

Definition 20. Summation Notation

All Riemann Sums are in the form,

$$\sum_{i=m}^{n} f(x_i) \cdot \Delta x$$

Left Riemann Sums are in the form,

$$\sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

Right Riemann Sums are in the form,

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

Method 13. Strategy for Finding Riemann Sums in Summation Notation

All Riemann Sums will be in the form,

$$\sum_{i=m}^{n} f(x_i) \cdot \Delta x$$

1. Find Δx, which is the width of the rectangles, usually $\frac{b-a}{n}$ where a and b are bounds of the integral and n is the number of rectangles
2. Find i, generally use $i = 0$ for left sums and $i = 1$ for right sums
3. Find $f(x_i)$ by plugging $x_i = a + i \cdot \Delta x$ into the function $f(x)$

Example 27

For $f(x) = x^2 - 1$, there are 9 equal subdivisions over the interval $[-10, -1]$. Represent this sum in summation notation form using a Right Riemann Sum.

1. Find Δx
$$\Delta x = \frac{-1 - (-10)}{9} = 1$$
2. Find m n for the sum. Since this is a Right Riemann Sum, $m = 1$
3. Find $f(x_i)$. Since $x_i = -10 + 1 \cdot i = i - 10$, $f(x_i) = (i - 10)^2 - 1$
4. Putting it all together, we get

$$\boxed{R(9) = \sum_{i=1}^{9} (i - 10)^2 - 1}$$

5.2 Definite Integrals

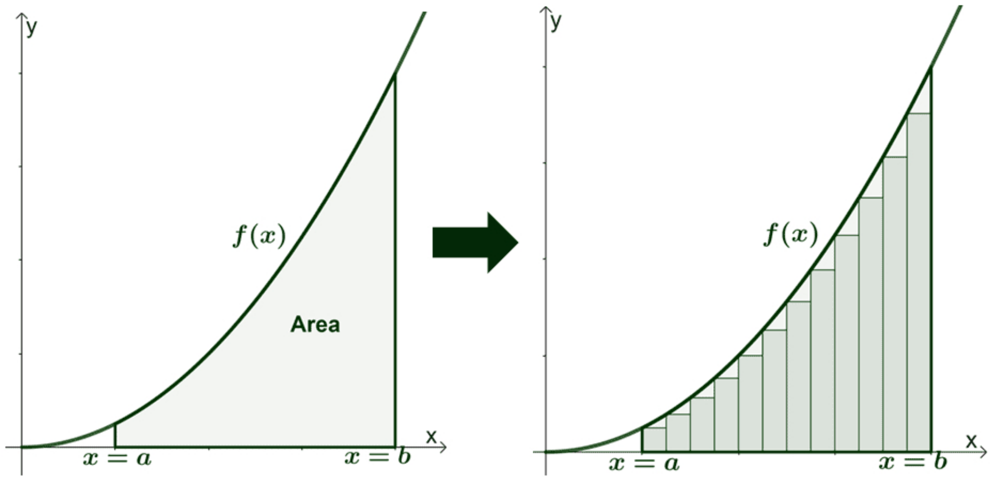
Definite integrals are a way to find the precise area under a curve. The definition shows that a definite integral is the sum of infinite tiny rectangles from a Riemann Sum under a curve, which can be used to approximate the area under a curve. A definite integral is written with two numbers, a and b, which define the start and end x coordinates of the area to be found.

Definition 21. Definite Integral as a Riemann Sum

The definite integral of a function $f(x)$ can be defined as the limit of a Riemann Sum,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x$$

In a definite integral, the parts on the top and the bottom of the integral are called the limits or the bounds of the integral, and the dx is called a differential. Differentials can be interpreted as tiny changes in the direction of x, or an extremely small Δx.



Example 28

Write $\lim_{n\to\infty} \sum_{i=1}^{n} \ln(2 + \frac{5i}{n}) \cdot \frac{5}{n}$ as a definite integral.

Taking apart the Riemann Sum representation of the definite integral, we see that $f(x_i) = \ln(2 + \frac{5i}{n})$ and $\Delta x = \frac{5}{n}$. We can also note that this definite integral starts at $x = 2$, and we have to find the end bound.

Since $\Delta x = \frac{b-2}{n} = \frac{5}{n}$. Solving for b, we get $b = 7$. Thus, the definite integral can be written as

$$\boxed{\int_2^7 \ln(x)dx}$$

5.3 Fundamental Theorem of Calculus Part 1

The first part of the Fundamental Theorem of Calculus relates the derivative and the integral as the antiderivative. Note that every continuous function f has an antiderivative F. If f is integrated, and then differentiated, the function f will be found.

Theorem 7. Fundamental Theorem of Calculus (1)

If $f(x)$ is continuous over $[a, b]$, and $a \leq x \leq b$, and

$$F(x) = \int_a^x f(t)dt$$

then

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)(dt)\right] = f(x)$$

Example 29

Find

$$\frac{d}{dx}\left[\int_\pi^{2x} \frac{\cos^2(t)}{\ln(t-\sqrt{t})}dt\right]$$

Use the chain rule to differentiate the function, treating the inner function as $2x$.

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_\pi^{2x} \frac{\cos^2(t)}{\ln(t-\sqrt{t})}$$

$$f(x) = \frac{\cos^2(2x)}{\ln(2x-\sqrt{2x})} \cdot \frac{d}{dx}[2x]$$

$$= \boxed{\frac{\cos^2(2x)}{\ln(2x-\sqrt{2x})} \cdot 2}$$

5.4 Properties of Definite Integrals

Definition 22. Properties of Definite Integrals

Sum and Difference/Constant Multiple:

$$\int_a^b c_1 f(x) \pm c_2 g(x)dx = c_1 \int\limits_a^b f(x)dx \pm c_2 \int_a^b g(x)dx$$

Reverse Interval:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Zero Length Interval:

$$\int_a^a f(x)dx = 0$$

Adding Intervals:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Example 30

Using the Fundamental Theorem of Calculus Part 1 (FTC 1), find $\frac{d}{dx}\left[\int\limits_x^{x^2} \frac{\cos(t)}{t}dt\right]$

First we use the addition of intervals property to separate the integral into two parts, each we can take the derivative of.

$$\frac{d}{dx}\left[\int_x^c \frac{\cos(t)}{t}dt + \int_c^{x^2} \frac{\cos(t)}{t}dt\right]$$

And then we use the sum property to take the derivative of each integral.

$$\frac{d}{dx}\left[\int_x^c \frac{\cos(t)}{t}dt\right] + \frac{d}{dx}\left[\int_c^{x^2} \frac{\cos(t)}{t}dt\right]$$

Next, we can rewrite the first integral so that the x is the upper limit by negating the integral.

$$-\frac{d}{dx}\left[\int_c^x \frac{\cos(t)}{t}dt\right] + \frac{d}{dx}\left[\int_c^{x^2} \frac{\cos(t)}{t}dt\right]$$

Now we can apply the Fundamental Theorem of Calculus Part 1 to differentiate the integrals.

$$\boxed{-\frac{\cos(x)}{x} + \frac{\cos(x^2)}{x^2} \cdot 2x}$$

5.5 Fundamental Theorem of Calculus Part 2

The second part of the Fundamental Theorem of Calculus describes a way to evaluate definite integrals using the antiderivative (the "reverse" derivative) of a function.

Definition 23. Antiderivative

A function $F(x)$ is the antiderivative of a function $f(x)$ if its derivative is $f(x)$, such that

$$f(x) = \frac{d}{dx}\left[F(x)\right]$$

Theorem 8. Fundamental Theorem of Calculus (2)

If $f(x)$ is continuous over $[a, b]$ and $F(x)$ is the antiderivative of $f(x)$ ($F'(x) = f(x)$), then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

which can also be written as

$$\int_a^b F'(x)dx = [F(x)]_a^b = F(b) - F(a)$$

5.6 Indefinite Integrals

It is important to note that antiderivatives are not unique, meaning a function can have many antiderivatives. This is because antiderivatives can differ by a constant, thus we can write the general antiderivative in the form $F(x) + C$ for some constant C. This can be shown in another type of integral, called an indefinite integral. Indefinite integrals find the general antiderivative of a function and do not have limits/bounds.

Definition 24. Indefinite Integral

An Indefinite Integral is an integral without bounds/limits, which finds the general antiderivative of a function. The antiderivative or indefinite integral of a function always has a constant to represent the infinitely many antiderivatives.

$$\int f(x) = F(x) + C$$

Definition 25. Properties of Indefinite Integrals

Sum and Difference/Constant Multiple:

$$\int c_1 f(x) \pm c_2 g(x)dx = c_1 \int f(x)dx \pm c_2 \int g(x)dx$$

The rest of the chapter is dedicated to methods and formulas used to find the antiderivatives, which can in turn be used to find the definite integrals of functions.

5.7 Reverse Power Rule

The reverse power rule is the inverse of the power rule when differentiating.

Definition 26. Reverse Power Rule

For function $f(x) = x^n$ where $n \neq -1$, the antiderivative $F(x)$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

for some constant C.

Example 31

Find $\int x^5 + x^3 + xdx$

$$\begin{aligned} \int x^5 + x^3 + xdx &= \int x^5 dx + \int x^3 dx + \int xdx \\ &= \frac{x^6}{6} + \frac{x^4}{4} + \frac{x^2}{2} + C \end{aligned}$$

To check, the resulting expression can be differentiated to check that it is the antiderivative.

5.8 Antiderivative Identities

The antiderivative identities are remarkably similar to the derivative counterparts, except they are reversed. These should be memorized.

Definition 27. Antiderivative Identities

1. $\int \frac{1}{x}dx = \ln|x| + C$
2. $\int \sin(x)dx = -\cos(x) + C$
3. $\int \cos(x)dx = -\sin(x) + C$
4. $\int \sec^2(x)dx = \tan(x) + C$
5. $\int \csc^2(x)dx = -\cot(x) + C$
6. $\int \sec(x)\tan(x)dx = \sec(x) + C$
7. $\int \csc(x)\cot(x)dx = -\csc(x) + C$
8. $\int e^x dx = e^x + C$
9. $\int a^x dx = \frac{a^x}{\ln(a)} + C$
10. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$
11. $\int \frac{1}{1+x^2}dx = \arctan(x) + C$

Example 32

Find $\int \frac{5}{x} + 2\sin(x)dx$

$$\begin{aligned}\int \frac{5}{x} + 2\sin(x)dx &= \int \frac{5}{x}dx + \int 2\sin(x)dx \\ &= 5\int \frac{1}{x}dx + 2\int \sin(x)dx \\ &= \boxed{5\ln|x| - 2\cos(x) + C}\end{aligned}$$

5.9 U-Substitution

U-substitution is a method that is inverse to that of the chain-rule in the differentiation chapter. It is able to be applied to integrals that contain a composition of functions, but will only be able to handle a specific type.

Method 14. U-Substitution

First, write the Integral in the form,

$$\int f(g(x)) \cdot g'(x)dx$$

Second, make a substitution $u = g(x)dx$ so that

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

Solve the antiderivative with respect to u, and substitute u back into the antiderivative to find the antiderivative in terms of x.

$\frac{du}{dx} = g'(x)$ can be solved for dx to plug it into the integral.

Example 33

Find $\int xe^{x^3+x^2} \cdot (3x + 2)dx$

First, write in the form $\int f(g(x)) \cdot g'(x)dx$, so

$$\int e^{x^3+x^2} \cdot (3x^2 + 2x)dx$$

Second, make the substitution $u = x^3 + x^2$, $du = 3x^2 + 2xdx$,

$$\int e^u du$$

Third, solve the antiderivative and substitute in $u = x^3 + x^2$.

$$\int e^u du = e^u + C = \boxed{e^{x^3+x^2} + C}$$

Example 34

Find $\int \sin(2x + 3)dx$

First, find the substitution to be used, which would be $u = 2x + 3$, thus $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$.

Plugging the above substitutions into the original integral and solving,

$$\int \sin(u) \cdot \frac{1}{2} du = -\frac{1}{2}\cos(u) + C = \boxed{-\frac{1}{2}\cos(2x+3) + C}$$

5.10 Integration by Parts

Integration by Parts is the integration equivalent of the product rule. It can find the antiderivative of a product of two functions.

Definition 28. Integration by Parts

Let $f(x)$ and $g(x)$ be continuous and differentiable functions. By integration by parts,

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

or using the substitutions $u = f(x)$ and $v = g(x)$

$$\int udv = uv - \int vdu$$

The integration by parts formula may need to be applied multiple times to find the antiderivative of a product of functions. The function we choose as u should only get simpler when differentiated, and the function we choose as v should not get more complicated when integrated.

Example 35

Find $\int x\cos(x)dx$

First, let $u = f(x) = x$ and $dv = g'(x)dx = \cos(x)dx$. Then $du = f'(x)dx = 1dx$ and $v = g(x) = \sin(x)$. Then by the integration by parts formula,

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)\cdot 1dx = \boxed{x\sin(x) + \cos(x) + C}$$

5.11 Integration with Partial Fractions

If there is a rational function with factors in the denominator where u-substitution doesn't appear to work initially, we can split the function into partial fractions to solve. By splitting the function into partial fractions, we are able to integrate much more easily.

Definition 29. Partial Fractions

For a rational function in the form $\frac{f(x)}{(x+a)(x+b)}$, the partial fraction decomposition of this function would be in the form

$$\frac{f(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

This method of decomposing a rational function into two partial fractions is also known as partial fraction decomposition.

Method 15. Partial Fraction Decomposition

To find $\int \frac{f(x)}{(x+a)(x+b)}dx$, we can decompose the rational functions into two partial fractions. First, we write the partial fractions in the form $\frac{A}{x+a} + \frac{B}{x+b}$. Then, solve the equation for A and B.

$$A(x+b) + B(x+a) = f(x)$$

Plugging the values back in, integrate each partial fraction by other methods or u-substitution.

$$\int \frac{A}{x+a}dx + \int \frac{B}{x+b}dx$$

Example 36

Find $\int \frac{x-5}{(2x-3)(x-1)}dx$.

When looking at the rational function, we can determine that the partial fractions will be in the form $\frac{A}{(2x-3)} + \frac{B}{x-1}$. Then, we solve the equation

$$A(x-1) + B(2x-3) = x-5$$
$$Ax - A + 2Bx - 3B = x-5$$

Thus by comparing the coefficients, $-A-3B=-5$, and $A+2B=1$. Solving these systems of equations, we get $A=-7$ and $B=4$. Plugging it back into the partial fractions, the integral now becomes

$$\int \frac{-7}{2x-3}dx + \int \frac{4}{x-1}dx = \boxed{-\frac{7}{2}\ln|2x-3| + 4\ln|x-1| + C}$$

5.12 Improper Integrals

Improper integrals are definite integrals that have either or both bounds/limits as $\pm\infty$, or an integrand that approaches ∞ at 1 or more points within its range.

Definition 30. Improper Integrals

If f is continuous over $[a, \infty)$ then,

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

If f is continuous over $(-\infty, a]$ then,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx$$

If f is continuous over $(-\infty, \infty)$ then,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx + \lim_{t\to\infty} \int_a^t f(x)dx$$

If the improper integral exists, the improper integral is **convergent**. If the improper integral doesn't exist, it is **divergent**.

Example 37

Find $\int\limits_1^{\infty} \frac{1}{x^2}dx$

$$\begin{aligned}\int\limits_1^{\infty} \frac{1}{x^2}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx \\ &= \lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t \\ &= \lim_{t\to\infty} \left[\frac{-1}{t} + 1\right] = \boxed{1}\end{aligned}$$

Chapter 6

Differential Equations

6.1 Introduction to Differential Equations

Definition 31. Differential Equation

A **differential equation** is an equation involving a function and one or more of its derivatives. The general solution to the differential equation is the function or set of functions $y = f(x)$ which satisfies the differential equation.

For example the equations listed below,

1. $y'' + 2y' = 3y$
2. $f''(x) + 2f'(x) = 3f(x)$
3. $\frac{d^2y}{dx} + 2\frac{dy}{dx} = 3y$

are all different representations of the same differential equation.

Definition 32. Initial Conditions/Particular Solutions

Initial Conditions to a differential equation are a set of conditions which describe the value of a solution at a point. Using the initial conditions, the **particular solution** can be found based on the initial conditions.

6.2 Slope Fields

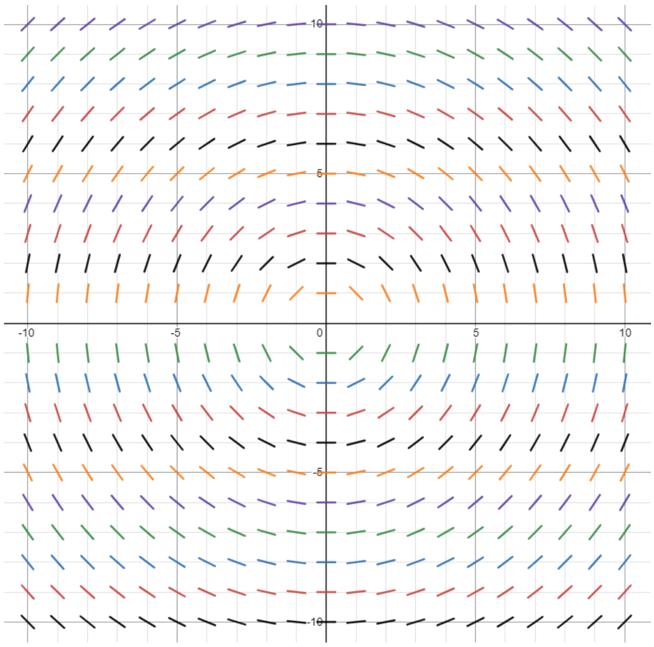
Slope fields are a collection of short line segments drawn on the coordinate plane that represents the slope of a function given by a differential equation. Solution curves can be traced using the slope curves as guidance.

Example 38

Find the slope field of $\frac{dy}{dx} = \frac{-x}{y}$ Using a table of values of x and y, we can determine some slopes in the coordinate plane.

x	y	$\frac{dy}{dx}$
0	1	0
1	1	-1
-1	-1	-1
1	-1	1

Using a few of these points, we can find the slope field to be like the slope field below, where each line segment at a point represents what $\frac{dy}{dx}$ is at (x, y).



6.3 Euler's Method

Euler's method is an iterative method used to solve particular solutions of differential equations using the property of local linearity. Given an initial point, the derivative or differential equation, and a specified Δx, we can estimate another point of the solution curve.

Method 16. Euler's Method

Given a derivative $\frac{dy}{dx}$, an initial point (x_1, y_1), and a determined step size Δx, the recursive formulas for Euler's method are

$$x_{i+1} = x_i + \Delta x$$

$$y_{i+1} = y_i + \frac{dy_i}{dx_i}\Delta x$$

Using a table is helpful for applying Euler's method.

Example 39

Use Euler's method with step size 0.1 to approximate $y(0.3)$ for initial value $y(0) = 1$ and equation $\frac{dy}{dx} = x + y$.

Using a table, we can apply Euler's method to find our desired value.

x	y	$\frac{dy}{dx}$
0	1	1
0.1	$1 + 1 \cdot 0.1 = 1.1$	1.2
0.2	$1.1 + 1.2 \cdot 0.1 = 1.22$	1.44
0.3	$1.22 + 1.44 \cdot 0.1 = 1.362$	N/A

Thus, $\boxed{y(0.3) \approx 1.362}$

6.4 Separable Differential Equations

separable differential equations are a type of differential equations which are able to be solved by separating the differential.

Definition 33. Separable Differential Equation

A separable differential equation is in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

or

$$g(y)dy = f(x)dx$$

Method 17. Solving Separable Differential Equations

To solve a differential equation in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (If not in the form, find a way to convert it):

1. Convert it into the form $g(y)dy = f(x)dx$ by multiplying both sides $g(y)$ and dx
2. Integrate both sides, so that $\int g(y)dy = \int f(x)dx$
3. Solve for y, and don't forget the $+C$

Example 40

Solve for y if $\frac{dy}{dx} = \frac{2x}{y}$.

First, convert the differential equation to the form listed above.

$$(y)dy = (2x)dx$$

Then integrate both sides,

$$\int (y)dy = \int (2x)dx$$

$$\frac{y^2}{2} = x^2 + C$$

Solving for y,

$$\boxed{y = \pm\sqrt{2x^2 + C}}$$

6.5 Exponential Models

Exponential models are differential equations where the rate at which y grows is proportional to the quantity of y.

Definition 34. Exponential Models

If y is a differentiable function of t, and

$$\frac{dy}{dt} = ky$$

then

$$y = Ce^{kt}$$

where C is the initial value, k is the proportionality constant.

When $k > 0$, the model represents exponential growth. When $k < 0$, the model represents exponential decay. An extension of exponential decay is Newton's Law of Cooling, which is defined below.

Theorem 9. Newton's Law of Cooling/Heating

If T is the temperature of an object at time t, and T_s is the temperature of the surrounding environment, then the rate of change of the temperature of the object can be modeled as

$$\frac{dT}{dt} = -k(T - T_s)$$

for some proportionality constant k.

Notice that by separation of variables and solving, the law of cooling can be rewritten as $T - T_s = (T_0 - T_s)e^{-kt}$, where T_0 is the temperature at time $t = 0$.

Example 41

Let T be the temperature in Fahrenheit of a potato in a room whose temperature is 60°. If the potato cools from 100° to 90° after being taken out of the oven for 10 minutes, how much longer will it take for its temperature to decrease to 80°?

We know that $T_s = 60$, $T_0 = 100$, and the temperature after 10 minutes is 90. Now we can set up a differential equation according to Newton's Law of Cooling and solve it.

$$\begin{aligned}
\frac{dT}{dt} &= k(60-T) \\
\frac{1}{60-T}dT &= kdt \\
-\ln|60-T| &= kt + C \\
60 - T &= Ce^{-kt} \\
T &= 60 - Ce^{-kt}
\end{aligned}$$

Next, we can solve for the constant C by using the initial temperature at $t = 0$, $T = 100$.

$$\begin{aligned}
100 &= 60 - C \\
C &= -40
\end{aligned}$$

Thus the cooling equation becomes

$$T = 60 + 40e^{-kt}$$

Next we can solve for the proportionality constant by using the next point of temperature, at $t = 10$, $T = 90$.

$$\begin{aligned}
90 &= 60 + 40e^{-10k} \\
\frac{3}{4} &= e^{-10k} \\
\frac{-1}{10}\ln(\frac{3}{4}) &= k
\end{aligned}$$

Using this value for k, our cooling equation now becomes,

$$T = 60 + 40e^{\frac{1}{10}\ln(\frac{3}{4})t}$$

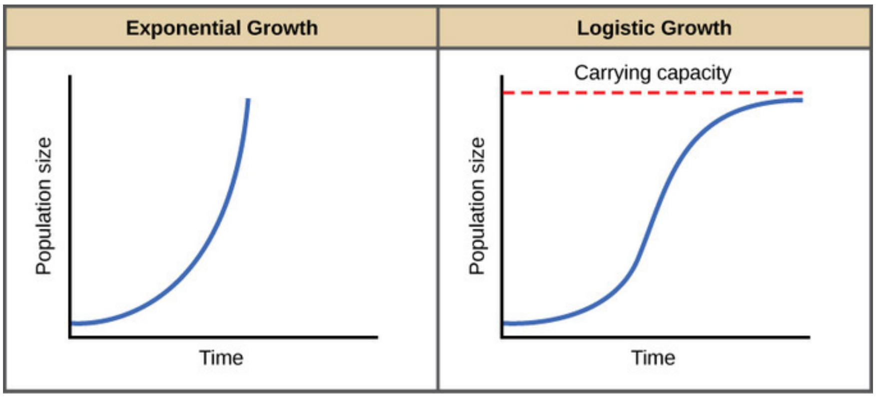
Now solving for our final value when the temperature will equal 80.

$$\begin{aligned}
80 &= 60 + 40e^{\frac{1}{10}\ln(\frac{3}{4})t} \\
\ln(\frac{1}{2}) &= \frac{3}{4}t \\
\frac{4}{3}e^{10\ln(\frac{1}{2})} &= t \\
t &\approx 24
\end{aligned}$$

6.6 Logistic Models

Logistic models are an extension of exponential models, but they asymptote toward an upper bound. These are frequently used to model population growth, where there is some certain limit to the population that it will asymptote towards known as the carrying capacity.

A comparison between an exponential and a logistic model is shown below.



Definition 35. Population Logistic Differential Equation

Given some population P dependent on time t, limiting or carrying capacity L, and some constant of proportionality k, the growth of population can be modeled as

$$\frac{dP}{dt} = kP(1 - \frac{P}{L})$$

Something else to note is that the fastest growth rate of the logistic model is at the inflection point, so find the second derivative of the population model and set it equal to 0. It is also the midpoint between the carrying capacity and the initial population, which can be solved for t.

Example 42

The population $P(t)$ of mice in a shed after t years can be modeled by $\frac{dP}{dt} = 3P \cdot \left(1 - \frac{P}{2500}\right)$. The initial population of mice is 1000. What is the population when it is growing the fastest?

To answer the first problem, the population at which it is growing the fastest is the midpoint between 1000 mice and the carrying capacity, 2500 mice. Thus, the population at which it is growing the fastest is $\boxed{1750}$ mice.

Chapter 7

Applications of Integration

7.1 Average Value of a Function

Definition 36. Average Value of a Function over an Interval

If f is continuous over interval $[a, b]$, the average value is

$$\text{average value} = \frac{\int_a^b f(x)dx}{b-a}$$

There is also a Mean Value Theorem for integrals which is much similar to the Mean Value Theorem for derivatives.

Theorem 10. Mean Value Theorem for Integrals

If f is continuous over $[a, b]$, then there exists a c where $a < c < b$ such that

$$f(c) = \frac{\int_a^b f(x)dx}{b-a}$$

The average value formula for the average value of a function is commonly used on the AP Calculus FRQ sections. Make sure that this formula is not confused with the average rate of change formula, which is $\frac{f(b)-f(a)}{b-a}$.

Example 43

From the year 2000 (t=0) to the year 2003 (t=3), the number of people who bought a chicken nugget from McDonalds in a certain year is modeled by the function $p(t) = e^{3t}$. Find the average number of customers per year over the span of 20 years.

We will use the formula for the average value of a function below.

$$\text{average value} = \frac{1}{3-0}\int_0^3 e^{3t}dt$$

Using u-substitution, we can evaluate this expression to be,

$$\text{average value} = \frac{1}{3}\left[\frac{1}{3}e^{3t}\right]_0^3$$

Thus, the average number of people is $\boxed{900.232}$.

7.2 Position, Velocity, Acceleration with Integrals

Since indefinite integrals represent antiderivatives, it makes sense that the relationship between position, velocity, and acceleration with integrals is the reverse direction. This can be summarized in the arrow chart below.

$$\text{Acceleration} \overset{\int}{\Longrightarrow} \text{Velocity} \overset{\int}{\Longrightarrow} \text{Position}$$

When given an acceleration/velocity function and are asked to find the velocity/position functions, the function will need to be integrated and satisfy initial conditions. This is an extension of solving differential equations with initial conditions.

Most commonly, the acceleration function is written with an $a(t)$, velocity function written with $v(t)$, and position function $s(t)$.

Definition 37. Displacement and Distance

Displacement: Given velocity function of a partial in rectilinear/one-dimensional motion $v(t)$, the integral from $t = a$ to $t = b$

$$\int_a^b v(t)dt$$

gives the displacement of the particle from $t = a$ to $t = b$. This is the difference between the final and the initial position.

Distance: Given velocity function of a partial in rectilinear/one-dimensional motion $v(t)$, the integral from $t = a$ to $t = b$

$$\int_a^b |v(t)|dt$$

gives the displacement of the particle from $t = a$ to $t = b$. This is the total distance the particle traveled between the two times. If the velocity function is positive, then the displacement equals the distance.

Example 44

Find the position function given acceleration function $a(t) = 9$ and initial conditions $v(0) = 3$ and $s(0) = 8$.

To find the position function, integrate twice while plugging in the initial conditions to find the arbitrary constants.

$$a(t) = 8$$
$$v(t) = \int 8dt$$
$$v(t) = 8t + C \implies v(0) = 3 = C \implies v(t) = 8t + 3$$
$$s(t) = \int 8t + 3dt$$
$$s(t) = 4t^2 + 3t + C \implies s(0) = 9 = C$$

Thus, the position function is $\boxed{s(t) = 4t^2 + 3t + 9}$

7.3 Area Between Curves

The area between two curves is usually the difference between the integrals of each of them between two bounds. However, when the curves cross each other, it gets slightly complicated with what to integrate.

Definition 38. Area Between Curves

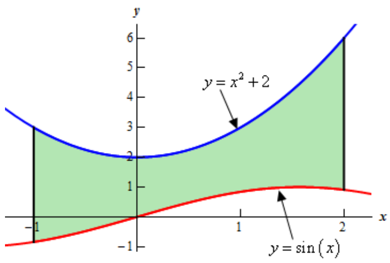
The area between two curves $f(x)$ and $g(x)$ over the interval $[a, b]$ where $f(x) \geq g(x)$ over the interval is,

$$\int_a^b f(x) - g(x)dx$$

When curves intersect, sometimes the functions need to be swapped depending on which is greater over the interval. If the functions to find the area between are functions of y, then integrate with respect to y using the y coordinate bounds.

Example 45

Find the area between $y = x^2 + 2$ and $y = \sin(x)$ from $x = -1$ to $x = 2$.



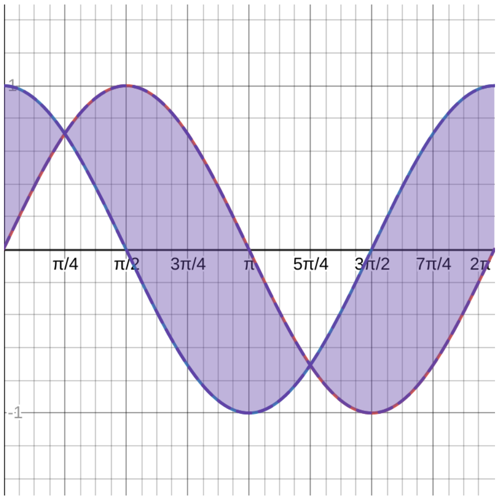
Based on the graph of the two functions by solving them analytically, we can see that $y = x^2 + 2$ is greater than $y = \sin(x)$ over this interval. Thus, we can integrate the difference over the whole interval, which is

$$\int_{-1}^{2} x^2 + 2 - \sin(x)dx = \left[x^3/3 + 2x + \cos(x)\right]_{-1}^{2} = 9 + \cos(2) - \cos(1) = \boxed{8.04355}$$

Example 46

Find the area between $y = \sin(x)$ and $y = \cos(x)$ from 0 to 2π.

Taking a look at the graph of the two functions, we can notice that they intersect at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, where $\cos(x)$ is greater in the first interval up to $x = \frac{\pi}{4}$, $\sin(x)$ is greater in the second interval up to $x = \frac{5\pi}{4}$, and in the third interval $\cos(x)$ is greater up to $x = 2\pi$.



Using these comparison, we can construct the integrals to be,

$$\text{Area} = \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x) - \cos(x)dx + \int_{\frac{5\pi}{4}}^{2\pi} \cos(x) - \sin(x)dx$$

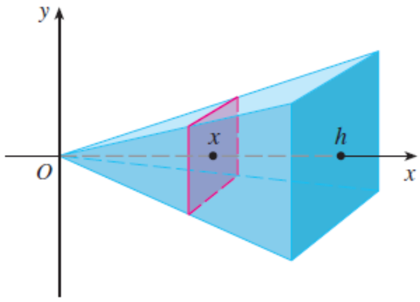
$$\text{Area} = [\sin(x) + \cos(x)]_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin(x) + \cos(x)]_{\frac{5\pi}{4}}^{2\pi}$$

$$\text{Area} = \left[\sqrt{(}2) - 1\right] + \left[\sqrt{2} - \sqrt{2}\right] + \left[1 + \sqrt{2}\right]$$

$$\text{Area} = 2\sqrt{2}$$

7.4 Volume with Cross Sections

Sometimes, we need to find the volume of various 3-D figures defined by using a 2-D plane as a cross-section.



In the above image, a 3-D solid drawn consists of a 2-D cross section. The point of this section is to show that the volume of the solid can be found by integrating over the bounds of the area of the cross section. This is analogous to piecing together all of the thin cross sections to form the 3-D figure.

Definition 39. Volume of Solids with Cross Sections

If the cross section of a figure perpendicular to the x-axis is given by $A(x)$ from points $x = a$ to $x = b$, then the volume of the solid is given by

$$V = \int_a^b A(x)dx$$

If the cross section of a figure perpendicular to the y-axis is given by $A(y)$ from points $y = a$ to $y = b$, then the volume of the solid is given by

$$V = \int_a^b A(y)dy$$

The use of the following formulas can be used for the area of a cross section. This should be a review from high school geometry.

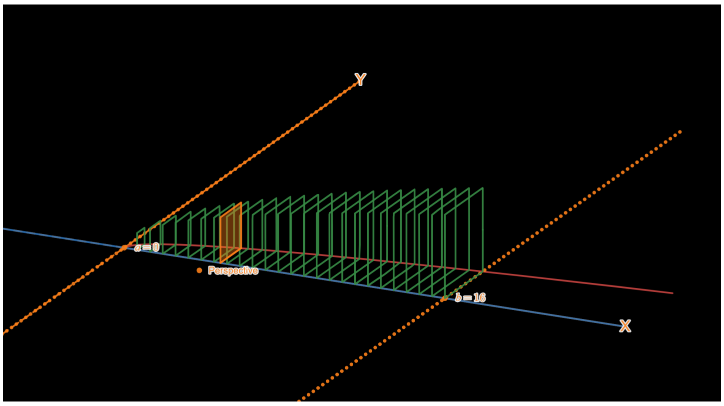
Square	s^2
Triangle	$\frac{1}{2}bh$
Equilateral Triangle	$\frac{\sqrt{3}}{3}s^2$
Circle	πr^2

The strategy to find volumes with known cross sections is to first draw the region of the base of a solid. Then, draw a single cross section with the correct orientation and find the formula for the area in terms of x or y. Then integrate with respect to the axis it is perpendicular to.

Example 47

Find the volume of the solid whose base is bounded by the curves $y = \sqrt{x}$ and $y = 0$ from $x = 0$ to $x = 16$, where the cross sections perpendicular to the x-axis are squares.

The solid being inquired is shown below.



Each cross section is the side-length of the square all squared, so the area function of each cross section would be,

$$A(x) = (\sqrt{x} - 0)^2 = x$$

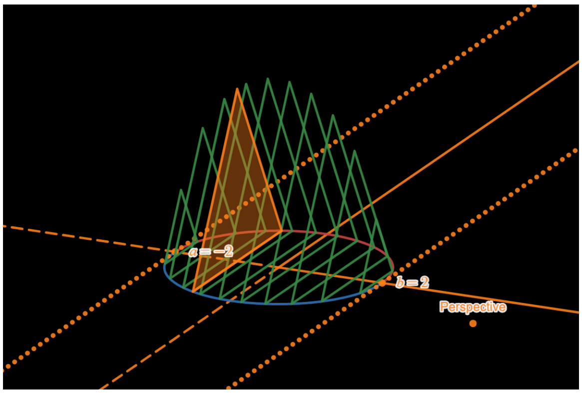
Now, integrating it between the bounds, we get

$$V = \int_0^{16} A(x)dx = \int_0^{16} xdx = \left[\frac{1}{2}x^2\right]_0^{16} = \frac{1}{2}(256 - 0) = \boxed{128}$$

Example 48

Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$, with cross sections perpendicular to the x-axis which are equilateral triangles.

The solid in question is shown below.



Each cross section is an equilateral triangle, so we know the formula to be used should be

$$A = \frac{\sqrt{3}}{4}s^2$$

To find the length of a side, we can rewrite the circle equation to the formula $y = \sqrt{4 - x^2}$. Since this is the length of the side from the x-axis to the point on the circle, we need to double this for the side length of each equilateral triangle cross section. Thus, $s = 2\sqrt{4 - x^2}$.

Then, the cross-section area function is

$$A(x) = \frac{\sqrt{3}}{4}\left(2\sqrt{4 - x^2}\right)^2 = 4\sqrt{3} - \sqrt{3}x^2$$

Now, we integrate it between the two bounds.

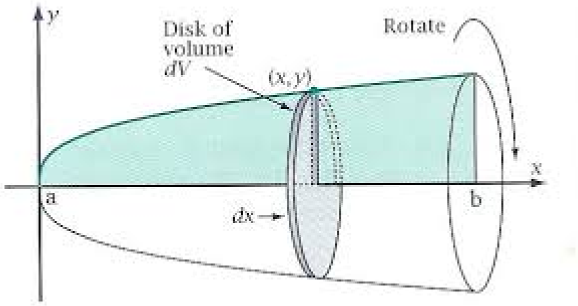
$$\int_{-2}^{2} A(x)dx = 2\int_0^2 4\sqrt{3} - \sqrt{3}x^2 dx = 2\left[4\sqrt{3}x - \frac{\sqrt{3}}{3}x^3\right]_0^2 = 2\left[8\sqrt{3} - \frac{8\sqrt{3}}{3}\right] = \boxed{\frac{32\sqrt{3}}{3}}$$

7.5 Volume of Solids of Revolution

Finding the solids of revolution is similar to finding the volume of a solid given the cross section. But instead, the cross section is a circle with the radius being the value of the function at the point. Sometimes, the cross section is called a disk because it resembles one.

It can be thought of that the graph of the function was revolved around the x or y-axis, and the solid if the volume enclosed by the revolution.

An example solid of revolution is shown below.



The general formula for finding the volume is the same as finding the volume of solids given cross sections, but the area of the cross section may be more specifically defined based on the scenario.

7.5.1 Volume with Disk Method

In the method for finding volumes with disk methods, the solid given will usually be of one function rotated around either a line parallel to the x or y-axis. Cross sections will be perpendicular to either the x or y-axis.

Method 18. Disk Method for Finding Volumes of Solids of Revolution

If the revolving region is bounded by $y = f(x)$ and the x-axis, and the region is revolved around the x-axis from $x = a$ to $x = b$, then the volume of the solid is

$$\int_a^b \pi \left[f(x)\right]^2 dx$$

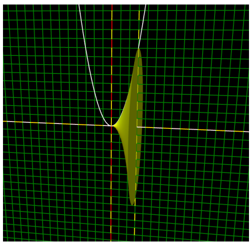
If the revolving region is bounded by $x = f(y)$ and the y-axis, and the region is revolved around the y-axis from $y = a$ to $y = b$, then the volume of the solid is

$$\int_a^b \pi \left[f(y)\right]^2 dy$$

Example 49

Find the volume of the solid where $f(x) = x^2$ is revolved around the x-axis from $x = 0$ to $x = 3$.

The solid in question is shown below.



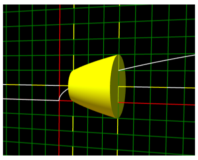
Knowing that each cross section is a circle where the radius is the height of the function from the rotated axis, we see that $A(x) = \pi(x^2)^2 = \pi x^4$. Now plugging this into the integral to find the volume, we get

$$\int_0^3 \pi x^4 dx = \pi \left[\frac{1}{5}x^5\right]_0^3 = \boxed{\frac{243\pi}{5}}$$

Example 50

Find the volume of the solid where $f(x) = \sqrt{x}$ is revolved around the line $y = 1$ from $x = 1$ to $x = 4$.

The solid in question is shown below.



Because the solid is revolved around the line $y = 1$, the radius of the cross section is $r = \sqrt{x} - 1$.

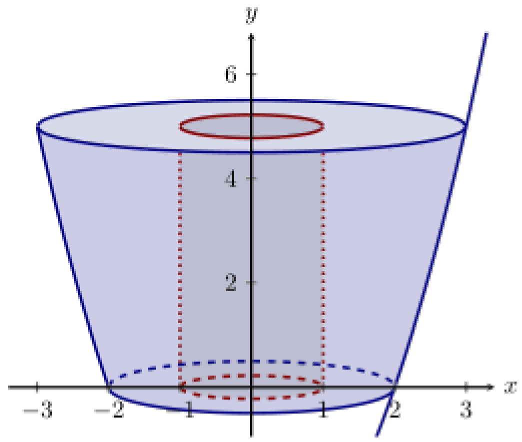
Plugging this into the formula for the solid of revolution, we get

$$\int_1^4 \pi \left[\sqrt(x) - 1\right]^2 dx = \pi \int_1^4 x - 2\sqrt{x} + 1 dx = \pi \left[\frac{x^2}{2} - \frac{4}{3}x^{\frac{3}{2}} + x\right]_1^4 = \boxed{\frac{-13\pi}{6}}$$

7.5.2 Volume with Washer Method

Finding the volume of solids of revolutions is used when the region revolved is bounded between two curves and revolved around another line parallel to the x or y-axis.

Each cross section is the difference between the circles created by the outer and the inner function, like a washer. An example solid where this method would be used is shown below.



Method 19. Washer Method for Finding Volumes of Solids of Revolution

If the revolving region is bounded by $y = f(x)$ and $y = g(x)$ over $[a, b]$ and revolved about the x-axis given $f(x) \geq g(x)$, then the volume is given by

$$\int_a^b \pi \left[f(x)^2 - g(x)^2\right] dx$$

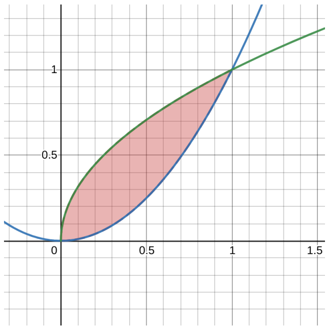
If the revolving region is bounded by $x = f(y)$ and $x = g(y)$ over $[a, b]$ and revolved about the y-axis given $f(y) \geq g(y)$, then the volume is given by

$$\int_a^b \pi \left[f(y)^2 - g(y)^2\right] dy$$

Example 51

Find the volume of the solid bounded by curves $f(x) = x^2$ and $g(x) = \sqrt{x}$ about the x-axis.

The area to be revolved around the x-axis is shown below.



Since the function $y = \sqrt{x}$ is greater than $y = x^2$, the area of each cross section would be

$$A(x) = \pi \left[\sqrt{x}^2 - (x^2)^2 \right] = \pi[x - x^4] = \pi x - \pi x^4$$

Integrating this along the interval where the region is bounded by from $x = 0$ to $x = 1$, we get

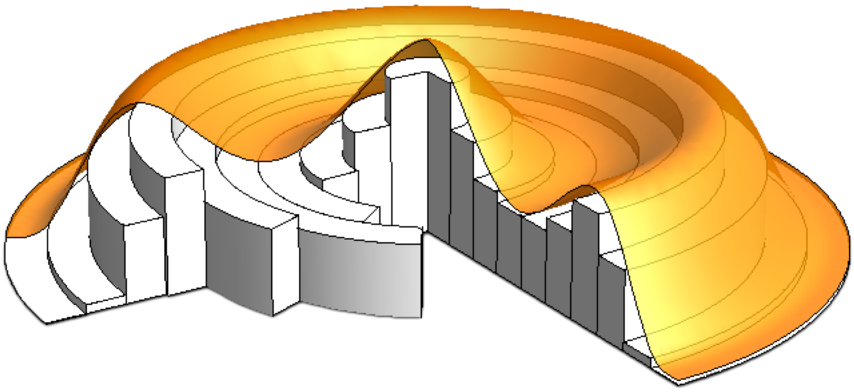
$$\int_0^1 \pi x - \pi x^4 dx = \left[\frac{\pi}{2} x^2 + \frac{\pi}{5} x^5 \right]_0^1 = \boxed{\frac{\pi}{2} + \frac{\pi}{5}}$$

7.5.3 Volume with Cylindrical Shells Method

In the past two sections, we have integrated the circular cross section in the direction it is perpendicular. However, we can use this method of cylindrical shells to integrate functions which revolve around the y-axis in the x-direction without needing to change it to a function of y.

Instead of using cross sections, we will use shells in a cylindrical shape. Each cylindrical shell will have height $f(x)$, thickness dx, radius x (where the y-axis is the axis of rotation) and volume $V = 2\pi x f(x)dx$. By integrating this volume function over the x-bounds, we can find the volume of the solid revolved around the y-axis.

An example of a cylindrical shell model is shown below.



Method 20. Cylindrical Shells Method for Finding Volumes of Solids of Revolution

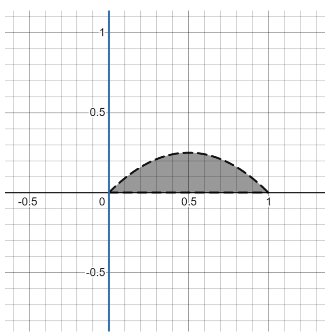
The volume of a solid obtained by rotating $y = f(x)$ about the y-axis from $x = a$ to $x = b$ where $0 \leq a < b$ is

$$V(x) = \int_a^b 2\pi x f(x)dx$$

Example 52

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

The region in question is shown below.

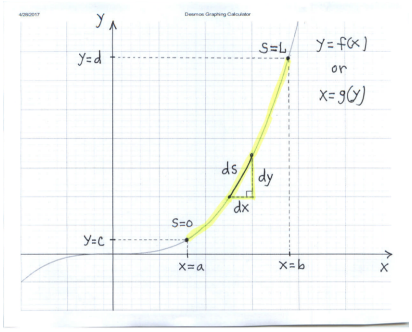


Since it is rotated about the y-axis, we know that the volume of each shell is $2\pi x(x - x^2)$. Then, we can integrate this between the bounds $x = 0$ and $x = 1$. The volume is then

$$\int_0^1 2\pi(x^2 - x^3)dx = 2\pi\left[\frac{1}{3}x^3 + \frac{1}{4}x^4\right]_0^1 = \boxed{\frac{\pi}{6}}$$

7.6 Arc Length

To find the arclength of a function, it can be thought that each small segment of a function's graph is some value dS, where it is composed of dx and dy components. Using the Pythagorean theorem, we can find that each $dS = \sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{dy}{dx}^2}$. A visual of this can be found below.



By integrating each dS, we can find the arclength of a function given two endpoints.

Definition 40. Arc-Length

If $y = f(x)$ and is continuous and differentiable over $[a, b]$, then the length L of $f(x)$ over the interval is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

If $x = f(y)$ is continuous and differentiable over $[a, b]$, then the length L of $f(y)$ over the interval is

$$L = \int_a^b \sqrt{1 + f'(y)^2} dy$$

Example 53

Find the arclength of the function $y = x^2$ from $x = 0$ to $x = 3$.

Plugging this into the formula for arc length, we get

$$\int_0^3 \sqrt{1 + 4x^2} dx = \boxed{9.747}$$

Chapter 8

Parametric Equations

8.1 Differentiating Parametric Functions

The derivative of a parametric equation can be interpreted as the slope of the tangent line to the parametric curve.

Definition 41. Derivative of Parametric Equation

Given parametric equation with functions $x = x(t)$ and $y = y(t)$, the derivative is defined as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Definition 42. 2nd Derivative of Parametric Equation

Given parametric equation with functions $x = x(t)$ and $y = y(t)$, the second derivative is defined as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

Example 54

Find the second derivative of the parametric equation $x = t^3 + 1$ and $y = t^4 - 4$.

First, we need to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{4t^3}{3t^2} = \frac{4}{3}t$$

Then, we plug into the second formula for the second derivative.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{4}{3}t\right]}{3t^2} = \frac{4}{9t^2}$$

Example 55

Find the slope of the tangent line to the parametric curve of $x = 3t$ and $y = 4t^2$ at point (3, 4).

Since the first derivative represents the slope of the tangent line at any point along the parametric curve, we find

$$\frac{dy}{dx} = \frac{8t}{3}$$

Then, we need to find the t-value where the point is located. By plugging the points into x and y, we find that $t = 1$. Plugging this into the derivative function, we get

$$\left.\frac{dy}{dx}\right|_{(3,4)} = \frac{8}{3}$$

8.2 Arc Length of Parametric Curves

Similar to the arc-length of regularly defined functions, parametric curves use the same idea of the Pythagorean theorem to construct a formula to evaluate the arc-length.

Definition 43. Arc Length of a Parametric Curve

If a parametric curve is defined by $x = x(t)$ and $y = y(t)$ from $t = a$ to $t = b$, then the arclength is defined as

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Something else to take note of is the speed of a particle that is defined along a parametric curve.

Definition 44

If a position parametric curve is defined for a particle by $x = x(t)$ and $y = y(t)$, then the speed of the particle at any given time is defined as

$$S(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

In other words, the arc-length of the parametric curve is the definite integral of the speed function.

Example 56

Find the arc-length of the parametric curve defined by $x = \sin(t)$ and $y = \cos(t)$ from $t = 0$ to $t = 8$.

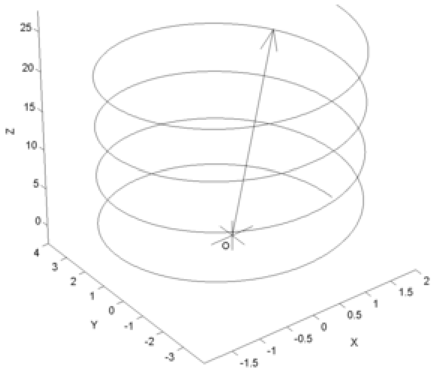
First, the derivatives of the parametric function need to be found. It can be found that $\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = -\sin(t)$. Plugging these into the arc-length formula, we get

$$L = \int_0^8 \sqrt{\sin^2(t) + \cos^2(t)}dt = \int_0^8 1dt = \boxed{8}$$

8.3 Vector-Valued Functions

Vector-valued functions are analogous to parametric equations, but instead of defining each value as a point, each value will be defined as a vector. Every t is mapped to a positional vector $\vec{r}(t)$.

An example of a vector-valued function is shown below.



Definition 45. Vector-Valued Function

Given two functions $x(t)$ and $y(t)$, the vector-valued function defined by these components is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} = \langle x(t), y(t)\rangle$$

The function $\vec{r}(t)$ represents the position of a particle. The derivative then represents the velocity, and the second derivative is the acceleration.

Definition 46. Differentiating Vector-Valued Functions

The derivative of the vector-valued function is defined as

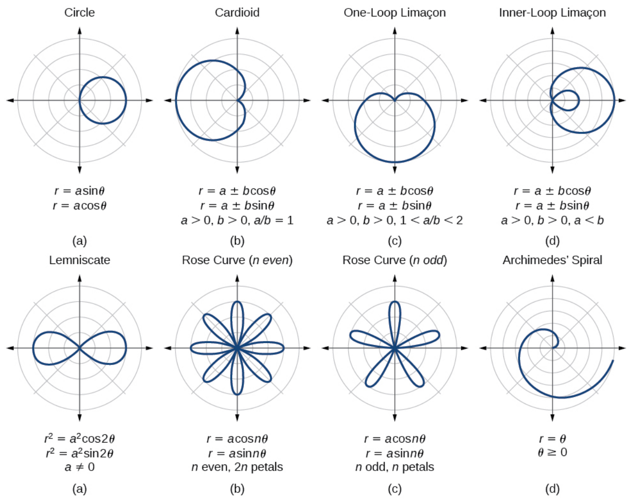
$$\vec{v}(t) = \lim_{h\to\infty} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = x'(t)\hat{i} + y'(t)\hat{j}$$

The second derivative of the vector-valued function is defined as

$$\vec{a}(t) = \lim_{h\to\infty} \frac{\vec{v}(t+h) - \vec{v}(t)}{h} = x''(t)\hat{i} + y''(t)\hat{j}$$

8.4 Differentiating Polar Coordinate Equations

Since polar coordinate equations are just parametric equations with the parameter function $r(\theta)$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$. Some examples of polar coordinate graphs are shown below.



Definition 47. Differentiating Polar Equations

Given polar function $r(\theta)$, the derivative is

$$\frac{dr}{d\theta} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

The derivative can be interpreted as the slope of the tangent line to the polar function at a point defined by r and θ .

Example 57

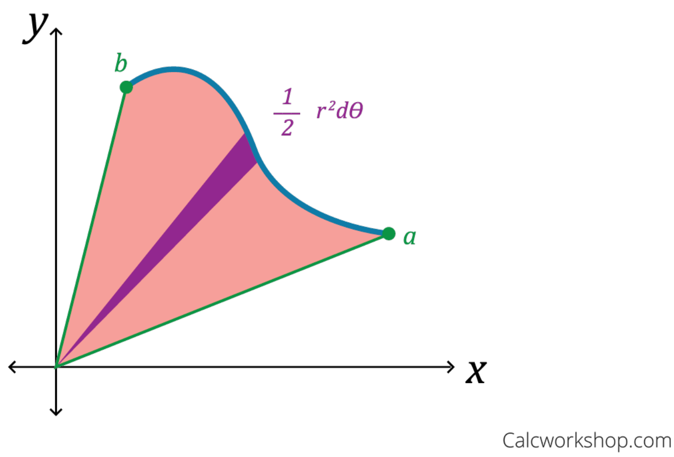
Find the derivative of the function $r(\theta) = 2$.

From this, we know that $x = 2\cos(\theta)$ and $y = 2\sin(\theta)$. We can take the derivatives of these to find that $\frac{dx}{d\theta} = -2\sin(\theta)$ and $\frac{dy}{dt} = 2\cos(\theta)$. Plugging these into the formula for the derivative, we get

$$\frac{dr}{d\theta} = \frac{-2\sin(\theta)}{2cos(\theta)} = -\tan(\theta)$$

8.5 Area with Polar Coordinates

The area enclosed within a polar coordinate function is found by integrating the area of small sectors of the polar graph, treating it like small sectors of a circle of radius $r = f(\theta)$. Thus, each sector would be of area $\frac{1}{2}r^2 d\theta$. A visual description is shown below.



Definition 48. Area of Polar Coordinate Function

Given a polar coordinate function $r = f(\theta)$, the area bounded by $\theta = \alpha$ and $\theta = \beta$ is

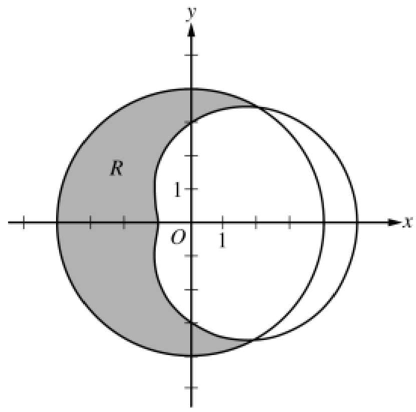
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$

Definition 49. Area Between Two Curves

Given function $r_0 = f(\theta)$ and $r_1 = g(\theta)$ where $f(\theta) \geq g(\theta)$, the area bounded by $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_1^2) d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left(f(\theta)^2 - g(\theta)^2 \right) d\theta$$

Example 58



The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos(\theta)$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. (AP Calculus BC 2018 5 (a))

Let R be the shaded region that is inside the graph $r = 4$ and also outside the graph of $r = 3 + 2\cos(\theta)$ as shown in the figure above. Write an expression involving an integral for the area of R.

Since R is bounded by $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$, we know these two are the bounds of the integral.

Also, since $r = 4$ is outside $r = 2\cos(\theta)$ over the interval, we know using the formula that $f(\theta) = 4$ and $g(\theta) = 2\cos(\theta)$. Plugging these all into the formula, we get

$$R = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{1}{2}\left(16 - 4\cos^2(\theta)\right) d\theta$$

Chapter 9

Sequences and Series

9.1 Geometric Series

Geometric series are series where each term is expressed as some number raised to some power. Compared to an arithmetic series, where each preceding term is a constant difference from the last, every term in a geometric series is a constant multiple from the last.

Definition 50. Geometric Series

Geometric series are any series that can be written in the form

$$\sum_{n=0}^{\infty} ar^n$$

Definition 51. Sum of Infinite Geometric Series

The sum of an infinite geometric series where $0 \leq r \leq 1$ is

$$S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Definition 52. Convergence of Geometric Series

The infinite geometric series defined by

$$\sum_{n=0}^{\infty} ar^n$$

converges for all r where $0 \leq r \leq 1$.

9.2 Harmonic Series and P-Series

The harmonic series is a series that is commonly used to determine convergence of a series.

Definition 53. Harmonic Series

The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

The p-series are generalized harmonic series to other degrees, where p is the degree the n in the denominator is raised to.

Definition 54. P-Series

The p-series defined by exponent p is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

Definition 55. Convergence of P-Series

Given a p-series defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

The series converges for all p where $p > 1$. This means that the harmonic series is divergent.

9.3 Convergence/Divergence Tests

There are six tests that test for divergence or convergence of an infinite series. They are listed below using the acronym DICLAR.

D Divergence Test

I Integral Test

C Comparison Test

L Limit Comparison Test

A Alternating Series Test

R Ratio Test

A flowchart helpful for navigating these tests can be found here.

9.3.1 Divergence Test

The Divergence Test tests if a series is divergent or not. If a series does not pass the divergence test, this does not mean it is convergent either, and another test must be applied.

Definition 56. Divergence Test

For some series defined by $\sum_{n=1}^{\infty} a_n$, if

$$\lim_{n\to\infty} a_n \neq 0$$

then the series will diverge.

Example 59

Does $\sum_{n=1}^{\infty} \frac{4n^2-n^3}{7-3n^3}$ diverge?

Using the divergence test, we need to find

$$\lim_{n\to\infty} \frac{4n^2 - n^3}{7 - 3n^3} = \frac{-n^3}{-3n^3} = \frac{1}{3}$$

Since this does not equal 0, the series is divergent.

9.3.2 Integral Test

The Integral Test uses the integral to determine if a series is convergent.

Definition 57. Integral Test

Suppose f(x) is a continuous, positive, and decreasing function over the interval $[k, \infty)$ and $f(n) = a_n$

1. If $\int\limits_k^\infty f(x)dx$ is convergent, then so is $\sum\limits_{n=1}^{\infty}$
2. If $\int\limits_k^\infty f(x)dx$ is divergent, then so is $\sum\limits_{n=1}^{\infty}$

Example 60

Determine if $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ is convergent or divergent.

Using the integral test, we will be comparing this series to the function $f(x) = \frac{1}{x \ln x}$. First, we need to find the improper integral of this function.

$$\int_1^\infty \frac{1}{x \ln x} dx = \lim_{t \to \infty} [\ln|\ln(x)|]_2^t = \infty$$

Thus, this series is $\boxed{\text{divergent}}$ by the Integral Test.

9.3.3 Comparison Test

The comparison test compares two series to determine if it is convergent or divergent. In simple terms, if a known series is convergent and another series is always less than the known series, then it is also convergent. The opposite applies for divergent series as well.

Definition 58. Comparison Test

Given two series $\sum a_n$ and $\sum b_n$, where both functions are non-negative, if $a_n \leq b_n$, then

1. If $\sum b_n$ is convergent, so is $\sum a_n$
2. If $\sum a_n$ is divergent, so is $\sum b_n$

Example 61

Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent or divergent using the Comparison Test.

A series similar to the given series that we know the convergence of is $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ which we know to be convergent because it is a p-series.

We also know that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is strictly less than $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

Based on these two assumptions, we can determine that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+1}$ is indeed $\boxed{\text{convergent}}$.

9.3.4 Limit Comparison Test

The limit comparison test uses similar conditions to the comparison test, but uses a limit to determine whether the series are convergent or not.

Definition 59

Given 2 series $\sum a_n$ and $\sum b_n$, where $a_n \geq 0$ and $b_n > 0$ define c so that

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If c is positive and finite, then either both series converge or both series diverge.

Example 62

Determine if $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ is convergent or divergent using the Limit Comparison Test.

We can compare this series to the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is convergent. Setting up the limit comparison test, we get

$$c = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \overset{\infty}{\lim_{n\to\infty}} \frac{2^n}{2^n-1} = 1$$

Since this value of c is positive and finite, this series is $\boxed{\text{convergent}}$.

9.3.5 Alternating Series Test

Alternating series are series whose terms alternate between being positive and negative. The convergence test for alternating series is much easier than the other, but make sure the conditions are met.

Definition 60. Alternating Series

An alternating series is in the form $\sum a_n$, where $b_n \geq 0$ and

$$a_n = (-1)^n b_n$$

or

$$a_n = (-1)^{n+1} b_n$$

There are only two conditions that need to be satisfied for an alternating series to be convergent.

Definition 61. Alternating Series Test

Given series $\sum a_n$ where $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ for all n, If

1. $\lim\limits_{n\to\infty} b_n = 0$
2. $|b_n|$ is a decreasing sequence

Then $\sum a_n$ is convergent.

Example 63

Determine if $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent or divergent.

First, recognize that $b_n = \frac{1}{n}$. Since this is always decreasing and $\lim\limits_{n\to\infty} b_n = 0$, the alternating series is indeed $\boxed{\text{convergent}}$.

9.3.6 Ratio and Root Test

The ratio test uses the common ratio between consecutive terms to determine convergence. This is similar to determining whether geometric series are convergent. These tests are helpful for series containing exponential functions and factorials.

Definition 62. Ratio Test

Given a series $\sum a_n$, let L be a number so that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then, if

1. L $<$ 1, the series is convergent
2. L $>$ 1, the series is divergent
3. L $=$ 1, the test is inconclusive

Definition 63. Root Test

Given $\sum a_n$, define L so that

$$L = \lim_{n\to\infty} \sqrt{n}|a_n| = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then, if

1. L $<$ 1, the series is convergent
2. L $>$ 1, the series is divergent
3. L $=$ 1, the test is inconclusive

Example 64

Determine if $\sum_{n=0}^{\infty} \frac{n!}{5^n}$ is convergent or divergent.

Using the ratio test, we need to first find L.

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{5} \right| = \infty$$

Since $\infty > 1$, this series is divergent.

9.4 Absolute/Conditional Convergence

Definition 64. Absolute/Conditional Convergence

Series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

Series $\sum a_n$ is conditionally convergent if $\sum |a_n|$ is divergent and $\sum a_n$ is convergent.

Note that if a series is absolutely convergent, it is also convergent. However, not all convergent series are also absolutely convergent.

Example 65

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is conditionally, absolutely, or not convergent at all.

First, we need to check the conditional convergence of the series. Since the alternating series is both decreasing and the limit to infinity is 0, the series is convergent.

Second, we need to check the convergence of the absolute value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is convergence because it is a p-series with $p > 1$.

This means that the series is $\boxed{\text{absolutely convergent}}$.

9.5 Alternating Series Remainder

The error or remainder of an approximation of an infinite alternating series using a select number of terms can be found with next value after the approximated terms. Since the value of the alternating series oscillates between the values after the final approximated term, we know that the error of the approximation could be at most the next term in the series.

Definition 65. Alternating Series Remainder

Given alternating series $\sum a_n = \sum (-1)^n b_n$ such that is satisfies the alternating series test, remainder R_n from the approximation of the series using n-terms, the value of the alternating series S, the approximated sum S_n,

$$|R_n| = |S - S_n| \leq |b_{n+1}|$$

Example 66

Find the maximum error that would result from estimating the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with 5 terms.

Since we approximate the series with 5 terms, the error or remainder would be bounded by the absolute value of the 6th term.

$$a_6 = \left| \frac{(-1)^{6+1}}{6} \right| = \frac{1}{6}$$

Thus, the remainder or error is

$$R_6 \leq \frac{1}{6}$$

9.6 Taylor/Maclaurin Polynomials

Taylor and Maclaurin polynomials are approximations of non-polynomial functions with polynomials. For example, the sine and cosine curves can be approximated with a series of polynomial terms. This involves the derivatives of the function centered at a certain point. If the approximation function is centered around 0, it is a maclaurin polynomial.

Definition 66. Taylor and Maclaurin Polynomials

For a function $f(x)$, an approximation of the function centered around $x = 0$ is called the Maclaurin Series/Polynomial and is defined as

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots$$

The approximation centered around $x = a$ is called the Taylor Series/Polynomial and is defined as

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f^{(3)}(0)}{3!} (x-a)^3 + \dots$$

Example 67

Find the Taylor Expansion of e^x at $x = 3$.

Since every derivative of the function e^x is e^x, we can easily substitute values into the formula for the Taylor Polynomial.

$$e^x \approx \sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n = e^3 + e^3(x-3) + \frac{e^3}{2!} (x-3)^2 + \frac{e^3}{3!} (x-3)^3 + \dots$$

9.6.1 Common Maclaurin Series

Some common Maclaurin Series, which are centered approximations about $x = 0$, are listed below. These should be memorized to make dealing with them easier.

Definition 67. Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots \text{for x in (-1, 1)}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{for all x}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{for all x}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{for all x}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \text{for x in (-1, 1]}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \text{for x in [-1, 1]}$$

9.6.2 Taylor Remainder Theorem/Lagrange Error Bound

Similar to finding the remainder or error bound of an alternating series, the error bound of a Taylor series can be found as well.

Definition 68. Taylor Remainder Theorem/Lagrange Error Bound

For the Taylor Series of $f(x)$ about $x = a$ with an estimation of n terms, the maximum error is given by

$$|R_n(x)| \leq \frac{\max\left|(f^{(n+1)}(z))\right|}{(n+1)!}(x-a)^{n+1}$$

where z is a value that maximizes the $n+1$ derivative of f(x) in the interval $[x, a]$.

Example 68

Find the maximum error of the approximation $\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$.

Note that this approximation uses 4 terms, so the remainder would be given by

$$|R_4(1.1)| \leq \frac{\max\left|(f^{(4+1)(z))}\right|}{(4+1)!}(1.1-1)^{4+1}$$

First, we need to find the fifth derivative of f(x) which is

$$f^{(5)}(x) = \frac{24}{x^5}$$

Then, to find the maximum $\max\left|\frac{24}{z^5}\right|$ over [1, 1.1] would be when 1 is plugged in, so the maximum error would be

$$|R_4(1.1)| \leq \frac{24}{(5)!}(0.1)^5 = \boxed{2 \cdot 10^{-6}}$$

9.7 Power Series

Power series are series that represent a function. Taylor and Maclaurin series both represent functions, which are power series.

Definition 69. Power Series

The power series for a function $f(x)$ about $x = c$ is defined as a series in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

The values given by a_n are called the coefficients of the power series.

For some functions, power series do not approximate for the entire domain of the function. At specific points, it may diverge from the actual function. The parts where the power series aligns with the function to be estimated is called the interval of convergence.

Definition 70. Interval of Convergence

The interval of convergence of a power series centered about $x = c$, where R is the radius of convergence is in the form

$$|x - c| < R$$

or

$$c - R < x < c + R$$

The points where $|x - c| = R$ need to be plugged into the function to determine if it converges or diverges at the point.

Method 21. Finding Intervals of Convergence

1. Using the Ratio Test, find the radius of convergence of the power series
2. Simplify the expression from above to find the interval of convergence
3. Test the points where $|x - c| = R$ to determine whether the power series converges at the two points

Example 69

Find the interval and radius of convergence of the power series
$f(x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n \cdot n}{4^n}(x+3)^n$

First, use the ratio test to find the radius of convergence. Let c be defined as

$$c = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot (n+1) \cdot (x+3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n \cdot n \cdot (x+3)^n} \right|$$

Simplified, we get

$$c = \lim_{n\to\infty} \left| \frac{-1 \cdot (n+1) \cdot (x+3)}{4 \cdot n} \right| = \left| \frac{(x+3)}{4} \right|$$

For this expression to be convergent, it has to be less than one, so

$$|x+3| < 4$$

From this, the radius of convergence can be determined to be 4.

Setting up the interval of convergence, we get

$$-7 < x < 1$$

But the points where $x = -7$ and $x = 1$ need to be tested for convergence.

At $x = -7$, the power series becomes $\sum_{n=1}^{\infty} n$, which is divergent.

At $x = 1$, the power series becomes $\sum_{n=1}^{\infty} (-1)^n \cdot n$, which is also divergent.

Thus, the interval of convergence is $\boxed{-7 < x < 1}$.

9.7.1 Differentiation and Integration of Power Series

Power series can be differentiated or integrated on a term-by-term basis. However, the intervals of convergence of the power series may change after.

Definition 71. Differentiation/Integration of Power Series

Given power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n(x-c)^n\right] = \sum_{n=0}^{\infty} n \cdot a_n(x-c)^{n-1}$$

$$\int \left[\sum_{n=0}^{\infty} a_n(x-c)^n\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot a_n(x-c)^{n+1}$$

Example 70

Find the power series for $\frac{1}{(x-1)^2}$ about $x = 0$.

First, note that this is the derivative of $\frac{1}{x-1}$. So we can take the derivative of the Maclaurin series to find the power series of this function.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{n=1}^{\infty} x^n\right] = \boxed{\sum_{n=1}^{\infty} nx^{n-1}}$$

9.8 Euler's Formula

Euler's formula is an important formula in mathematics that relates imaginary and real numbers. It can be derived from the combination of the sine and cosine Taylor series to form the e^x taylor series.

Definition 72. Euler's Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

Euler's Identity can be derived if the value $x = \pi$ is plugged into the Euler's Formula.

Definition 73. Euler's Identity

$$e^{i\pi} = -1$$